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EDITED BY

W. ALLEN WHITWORTH, M.A.,
FELLOW OF ST. JOHN'S COLLEGE, CAMBRIDGE.

C. TAYLOR, M.A.,
FELLOW OF ST. JOHN'S COLLEGE, CAMBRIDGE.

R. PENDLEBURY, M.A.,
FELLOW OF ST. JOHN'S COLLEGE, CAMBRIDGE.

J. W. L. GLAISHER, M.A., F.R.S.,
FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

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MESSENGER OF MATHEMATICS.

ON SPHERICAL HARMONICS.

By *W. D. Niven, M.A.*, Trinity College.

THE object in view in these articles is to give a method of deducing from what is known as the zonal harmonic of any degree the most general harmonic of the same degree.

As the method was suggested by Clerk Maxwell's treatment of the subject, and as the results here proved have a close alliance with those contained in his work on *Electricity and Magnetism*, the notation there used has been as far as possible adopted here.

After establishing a certain theorem on which the method above referred to depends, we will employ it in finding the most general expression for a harmonic referred to its poles as axes. We will then show its application in the case of symmetrical harmonics other than zonal. The last part of the paper will be taken up with a sketch of Professor Maxwell's proof of Laplace's expansion modified to some extent, especially by the insertion of the same theorem.

§1. The general expression given in the Electricity and Magnetism for a harmonic of the i^{th} degree is

$$(-1)^i M_i \frac{d^i}{dh_1, dh_2 \dots dh_i} \frac{1}{r},$$

where any operator $\dfrac{d}{dh}$ is the same as $l\dfrac{d}{dx} + m\dfrac{d}{dy} + n\dfrac{d}{dz}$.

It will thus be seen that the expression contains $2i+1$ independent constants, viz. one from M_i and $2i$ from the i arbitrary directions of the h-axes, each set of direction-cosines being of course connected by a relation of the form $l^2+m^2+n^2=1$. It may, conversely, be proved that any harmonic of the i^{th} degree can be thrown into the above form in only one way, so as to have all the quantities l, m, n real; in other words, that the i poles of the harmonic may be real points.*

If all the axes coincide, we have

$$(-1)^i \left(\frac{d}{dz}\right)^i \frac{1}{r},$$

and it is easy to prove that this is equal to

$$\frac{\lfloor i \, Q_i}{r^{i+1}},$$

where Q_i is a certain function of the polar coordinate θ. The quantity Q_i is called the zonal harmonic of the i^{th} degree, and may be easily obtained in various forms, two of which as being suitable to our purpose, we here quote

$$Q_i = \Sigma_n \left\{(-1)^n \frac{\lfloor 2i-2n}{2^i \lfloor n \lfloor i-n \lfloor i-2n} \mu^{i-2n}\right\} \ldots\ldots (A),$$

or

$$\Sigma_n \left\{(-1)^n \frac{\lfloor i}{2^{2n} \lfloor n \lfloor n \lfloor i-2n} \mu^{i-2n} \nu^{2n}\right\} \ldots\ldots\ldots (B),$$

μ being put for $\cos\theta$ and ν for $\sin\theta$.

It will be convenient to have an abbreviation for the operator belonging to the general harmonic, we therefore put

$$\frac{d^i}{dh_1 \ldots dh_i} = \left[\frac{d}{dh}\right]^i.$$

In harmony with the above relation between a harmonic and the corresponding spherical harmonic when it is zonal, we may put

$$(-1)^i \left[\frac{d}{dh}\right]^i \frac{1}{r} = \frac{\lfloor i \, Y_i}{r^{i+1}}.$$

The quantity Y_i is in general a function of the two polar coordinates θ and ϕ, it is the quantity we want to determine.

---

* Vide *Phil. Mag.*, October, 1876. Note on "Spherical Harmonics," by Professor Sylvester.

§ 2. We will now prove the theorem expressed by the equation

$$(-1)^i \left[\frac{d}{dh}\right]^i \frac{1}{r} = \left[\frac{d}{dh}\right]^i (r^i Q_i).$$

Let Q (fig. 1) be any point on a sphere of radius a, and let P be any other point; r, r' the distances of P from O, Q; also let $PQO = \pi - \theta$. Then

$$\frac{1}{r} = \frac{1}{a} - \frac{r'}{a^2} Q_1 + \ldots + (-1)^i \frac{r'^i}{a^{i+1}} Q_i + \ldots.$$

Let us now perform the operation $(-1) \left[\frac{d}{dh}\right]^i$ on both sides of this equation. If a, b, c be the coordinates of Q, we may write $x = \text{a} + x'$, &c. Hence

$$\begin{aligned}\frac{d}{dh} &= l \frac{d}{dx} + m \frac{d}{dy} + n \frac{d}{dz} \\ &= l \frac{d}{dx'} + m \frac{d}{dy'} + n \frac{d}{dz'} \\ &= \frac{d}{dh'};\end{aligned}$$

therefore $$(-1) \left[\frac{d}{dh}\right]^i = (-1) \left[\frac{d}{dh'}\right]^i.$$

Now $r'^n Q_n$ is a homogeneous function of the n^{th} degree of the coordinates x', y', z'. Hence, the effect of the operation $(-1)^i \left[\frac{d}{dh'}\right]^i$ upon the first i terms is to differentiate them down to zero. If after the differentiations have been performed on the other terms, we also put x', y', z' zero, we shall have only one term left, so that when $r' = 0$, or $r = a$,

$$(-1) \left[\frac{d}{dh}\right]^i \frac{1}{r} = \frac{1}{a^{i+1}} \left[\frac{d}{dh'}\right]^i (r'^i Q_i).$$

Hence, dropping the dashes on the right-hand side, we get .

$$\lfloor i \, Y_i = \left[\frac{d}{dh}\right] (r^i Q_i),$$

and $$(-1) \left[\frac{d}{dh}\right]^i \frac{1}{r} = \frac{1}{r^{i+1}} \left[\frac{d}{dh}\right]^i (r^i Q_i).$$

§ 3. In using this theorem to find the most general expression for a harmonic, since the choice of coordinates is arbitrary, we shall suppose that the axis of z passes through the point Q (fig. § 2). In that case, using the B formula for Q_i, we get

$$r^i Q_i = z^i - \frac{i(i-1)}{1.2} z^{i-2} \xi\eta + \ldots + (-1)^n \frac{\lfloor i}{2^{2n} \lfloor n \lfloor n \lfloor i-2n} z^{i-2n} (\xi\eta)^n + \ldots,$$

where

$$\xi = x + yj,$$
$$\eta = x - yj.$$

Any operator $$\frac{d}{dh} = l\frac{d}{dx} + m\frac{d}{dy} + n\frac{d}{dz},$$

becomes

$$\mathfrak{l}\frac{d}{d\xi} + \mathfrak{m}\frac{d}{d\eta} + n\frac{d}{dz},$$

where

$$\mathfrak{l} = l + mj,$$
$$\mathfrak{m} = l - mj;$$

and the angle $\cos^{-1}\mu_{pq}$ between two axes (l_p, m_p, n_p), (l_q, m_q, n_q), is given by

$$\mu_{pq} = \frac{\mathfrak{l}_p\mathfrak{m}_q + \mathfrak{l}_q\mathfrak{m}_p}{2} + n_p n_q.$$

The general operator on $Q_i r^i$, may be written

$$\left(\mathfrak{l}_1\frac{d}{d\xi} + \mathfrak{m}_1\frac{d}{d\eta} + n_1\frac{d}{dz}\right)\ldots\left(\mathfrak{l}_i\frac{d}{d\xi} + \mathfrak{m}_i\frac{d}{d\eta} + n_i\frac{d}{dz}\right).$$

Its effect on the general term of the expansion of $Q_i r^i$ is to produce a series of coefficients of which a specimen would be

$$(-1)^n \frac{\lfloor i}{2^{2n}} n_{i-2n+1}\, n_{i-2n+2} \ldots n_i V,$$

where V is a function of the $\mathfrak{l}$'s and $\mathfrak{m}$'s derived by picking out the coefficient of $\left(\frac{d}{d\xi}\right)^n \left(\frac{d}{d\eta}\right)^n$ from the factors

$$\left(\mathfrak{l}_1\frac{d}{d\xi} + \mathfrak{m}_1\frac{d}{d\eta}\right)\ldots\left(\mathfrak{l}_{2n}\frac{d}{d\xi} + \mathfrak{m}_{2n}\frac{d}{d\eta}\right).$$

It is obvious that any one of the terms of V will contain n $\mathfrak{l}$'s and n $\mathfrak{m}$'s, and the number of the terms will be

$$\frac{\lfloor 2n}{(\lfloor n)^2}.$$

Now the symmetry of V shows us that it is possible to express it in a series of terms of n products of the form

$$\mathfrak{l}_p\mathfrak{m}_q + \mathfrak{l}_q\mathfrak{m}_p,$$

where p and q may be any numbers less than n. But if we have $2n$ quantities, the number of factors that we can form of them, the sum of two of the quantities constituting a factor, taking n factors at a time is

$$\frac{\underline{|2n}}{2^n\,\underline{|n}},$$

equal to f suppose.

If now we arrange in lines the results of multiplying out each of these combinations of n factors we shall have 2^n individual terms on the right in each line. Hence, the sum of all the combinations of factors will produce $2^n f$ individual terms, *i.e.* $\dfrac{\underline{|2n}}{\underline{|n}}$. Hence, since the number of terms in V is this number, divided by $\underline{|n}$, it is clear that the symmetry is such that each individual term in the addition referred to is repeated $\underline{|n}$ times. It follows that $\underline{|n}\,V$ may be found by taking the combinations of factors as described. But we proved

$$\mathfrak{l}_p\mathfrak{m}_q + \mathfrak{l}_q\mathfrak{m}_p = 2\,(\mu_{pq} - n_p n_q);$$

therefore $\underline{|n}\,V = 2^n\,\{(\mu_{12} - n_1 n_2)\ldots(\mu_{1,\,n+1} - n_1 n_{n+1}) + \&\text{c.}\}$,

where all the changes must be rung among the factors.

All the changes amongst the n's in the general expression must also be taken.

The first three terms of the spherical harmonic of the i^{th} degree are

$$n_1 n_2 \ldots n_i - \tfrac{1}{2}\,[n_1 n_2 \ldots n_{i-2}\,(\mu_{(i-1)i} - n_{(i-1)} n_i) + \&\text{c.}]$$
$$+ \tfrac{1}{8}\,[n_1 n_2 \ldots n_{i-4}((\mu_{(i-3)(i-2)} - n_{i-3} n_{i-2})(\mu_{(i-1)i} - n_{i-1} n_i) + \&\text{c.}) + \ldots] + \&\text{c.}$$

The most general spherical harmonics of the 2nd, 3rd, and 4th degrees are easily derived by the direct application of the above method, but they may be deduced from the general formula now found. They are given by equating $\underline{|2}\,Y_2$, $\underline{|3}\,Y_3$, $\underline{|4}\,Y_4$ respectively to

$$3n_1 n_2 - \mu_{12},$$

$$3\,(5n_1 n_2 n_3 - \mu_{12} n_3 - \mu_{13} n_2 - \mu_{23} n_1),$$

$$3\,\{35 n_1 n_2 n_3 n_4 - 5\,(n_1 n_2 \mu_{34} + 5 \text{ similar terms})$$
$$+ \mu_{12}\mu_{34} + 2 \text{ similar terms}\}.$$

§4. As an example of the use of the general theorem of §2, in cases where the axes of coordinates are not arbitrary, we may take the case when σ of the axes of differentiation are in the equator and $i-\sigma$ coincide with the axes of z. Professor Maxwell points out that in this case*

$$\left[\frac{d}{dh}\right]^i = \left(\frac{d}{dz}\right)^{i-\sigma} \left\{\left(\frac{d}{d\xi}\right)^\sigma + \left(\frac{d}{d\eta}\right)^\sigma\right\}.$$

Let (α, β) be the polar coordinates of the point Q on the sphere (fig. 1, §2). Then

$$\mu = \cos\theta \cos\alpha + \sin\alpha \sin\theta \cos(\phi - \beta).$$

Hence, taking the first expression quoted for the zonal harmonic

$$r^i Q_i = \Sigma_n \left[(-1)^n \frac{\lfloor 2i-2n}{2^i \lfloor n \lfloor i-n \lfloor i-2n} \times \left\{\cos\alpha . z + \frac{\sin\alpha}{2}(\xi e^{-\beta} + \eta e^{\beta})\right\}^{i-2n} (z^2 + \xi\eta)^n\right].$$

Picking out the terms $z^{i-\sigma}\xi^\sigma$ and $z^{i-\sigma}\eta^\sigma$, and performing the required operation on them, we get

$$\frac{\cos\sigma\beta \sin^\sigma\alpha}{2^{i+\sigma-1}} \Sigma(-1)^n \frac{\lfloor 2i-2n \lfloor i-\sigma}{\lfloor n \lfloor i-n \lfloor i-2n-\sigma} \cos^{i-2n-\sigma}\alpha$$

$$= \frac{\lfloor 2i}{\lfloor i} \frac{\cos\sigma\beta \sin^\sigma\alpha}{2^{i+\sigma-1}} \left\{\cos^{i-\sigma}\alpha - \frac{(i-\sigma)(i-\sigma-1)}{2(2i-1)} \cos^{i-\sigma-2}\alpha + \ldots\right\}.$$

If we take

$$\left[A\left\{\left(\frac{d}{d\xi}\right)^\sigma + \left(\frac{d}{d\eta}\right)^\sigma\right\} + Bj\left\{\left(\frac{d}{d\xi}\right)^\sigma - \left(\frac{d}{d\eta}\right)^\sigma\right\}\right]\left(\frac{d}{dz}\right)^{i-\sigma}$$

---

* If, in factorizing any homogeneous operator, quadratic factors are introduced of the form

$$\left(A\frac{d}{dx} + B\frac{d}{dy} + C\frac{d}{dz}\right)^2 + \left(a\frac{d}{dx} + b\frac{d}{dy} + c\frac{d}{dz}\right)^2,$$

we see by throwing this into impossible factors that $A + ja$, &c. must be proportional to direction cosines. Hence $Aa + Bb + Cc = 0$. If, therefore, we put $K = \sqrt{(A^2 + B^2 + C^2)}$, $k = \sqrt{(a^2 + b^2 + c^2)}$, and suppose $K > k$, the operator becomes $K^2\left(\frac{d}{dX}\right)^2 + k^2\left(\frac{d}{dY}\right)^2$, where $\frac{d}{dX}$, $\frac{d}{dY}$ denote differentiations with regard to two lines at right angles. Hence the operator becomes, by Laplace's equation,

$$(K^2 - k^2)\left(\frac{d}{dX}\right)^2 - k^2\left(\frac{d}{dZ}\right)^2,$$

and the two real poles belonging to it are thus discovered.

as operator, where A and B are arbitrary constants, we thus arrive at Laplace's formula, given in Thomson and Tait, p. 149, (37), (38).

§ 5. In some of the expansions which occur it is necessary to find

$$\left[\frac{d}{dh}\right]^i \frac{1}{r^{2n+1}}.$$

In these cases it may be useful to observe that

$$\{1.3.5\ldots(2n-1)\}\left[\frac{d}{dh}\right]^i \frac{1}{r^{2n+1}} = (-1)^i \frac{1}{r^{2n+2i+1}}\left[\frac{d}{dh}\right]^i \left(r^i \frac{d^n Q_{n+i}}{d\mu^n}\right),$$

a theorem which can be proved in a similar manner to that contained in § 2. It is, however, quite as convenient to deduce $\dfrac{d^n Q_{n+i}}{d\mu^n}$ from the expansion of $(a^2 + 2ar\mu + \mu^2)^{-\frac{2n+1}{2}}$ by picking out the coefficient of μ^i as to differentiate Q_{n+i}.

LAPLACE'S EXPANSION.

§ 6. Consider a solid globe of homogeneous matter, the particles of which attract an outside particle with a force varying inversely as the square of the distance. Then, if the centre of the globe is taken for origin of coordinates, the potential at any outside point is $-\dfrac{M}{r}$, where M is the mass of the globe and r the distance of the point from its centre.

If another globe of equal dimensions and mass, but whose particles repel according to the inverse square of the distance, have its centre at a small distance h_1 from the origin along the line whose direction cosines are l_1, m_1, n_1; then the potential due to this globe is

$$M\left(\frac{1}{r} - h_1 \frac{d}{dh_1}\frac{1}{r}\right).$$

If we now imagine the two globes to coexist, then so far as attraction or repulsion is concerned we may suppose all that part of each which occupies common space to be annihilated. We thus arrive at the conception of a shell of matter, part of which attracts and part repels. The potential due to it is

$$-Mh_1 \frac{d}{dh_1}\frac{1}{r}.$$

Let the mass M now become infinitely large whilst the thick-

ness h_1 becomes infinitely small, the quantity Mh_1 remaining finite and equal to M_1. The potential due to the shell is then

$$-M_1 \frac{d}{dh_1} \frac{1}{r}.$$

Moreover, each of the spheres attracts or repels external matter as if its mass were collected at its centre. It is clear therefore that the shell will possess the same property. In Professor Maxwell's nomenclature this centre is then a compound point of the first degree. In accordance with this description we may call the shell a compound shell of the first degree.

Let us now suppose a compound shell of the first degree to be placed with its centre at the origin and let its potential be

$$M_1 \frac{d}{dh_1} \frac{1}{r}.$$

If we place with its centre at the point (l_2h_2, m_2h_2, n_2h_2), a shell exactly equal in all respects except that it repels where the other attracts and *vice versâ*, its potential, if h_2 is small, will be

$$-M_1 \left(\frac{d}{dh_1} \frac{1}{r} - h_2 \frac{d^2}{dh_1\, dh_2} \frac{1}{r}\right).$$

Let the two shells co-exist and let M_1h_2 be ultimately M_2, M_1 becoming infinite and h_2 zero; then the potential due to a compound shell of the second degree is

$$M_2 \frac{d^2}{dh_1\, dh_2} \frac{1}{r}.$$

Pursuing this reasoning, we might show that for a compound shell of the i^{th} degree, the potential at outside points is

$$(-1)^i M_i \frac{d^i}{dh_1\, dh_2 \ldots dh_i} \frac{1}{r}.$$

We have thus arrived at a physical meaning of a spherical harmonic, viz. that it is the potential of an infinitely thin shell of matter, part of which attracts and part repels according to a complicated arrangement determined by the positions of 2^i equal spheres whose centres ultimately coincide with the origin.

Since the action of the shell on outside matter is the same as that of a compound point at its centre, conversely outside matter will attract or repel the shell as if it were

concentrated at its centre. Hence, if it be placed in a field of force whose potential at the centre is V, the potential energy of the shell will be

$$M_i \left[\frac{d}{dh}\right]^i V,$$

(see Clerk Maxwell's *Electricity*, vol. I., p. 168).

For example, let this shell be surrounded by another which is concentric with it and produces a potential at inside points equal to

$$\left(\frac{r}{a}\right)^j Q_j.$$

The potential energy of the inner shell is then

$$\frac{M_i}{a^j}\left[\frac{d}{dh}\right]^i r^j Q_j,$$

being estimated on the compound point at the centre, and, therefore, we are to put $r=0$ after differentiation. But in those circumstances it is obvious the result is zero except when $i=j$, and we have proved, § 2, that in that case the result will be

$$\lfloor i \frac{M_i}{a^i} Y_{ij},$$

where Y_{ij} is the value of Y_i at the pole of the harmonics. But if σ_i is the symbol for the density of the inner shell at any point, then the expression for the potential energy estimated over the surface is

$$\iint \sigma_i Q_j ds.$$

This therefore will be zero except when $i=j$, and then it is

$$\frac{\lfloor i \, M_i}{a^i} Y_{ij}.$$

Let us now draw a line from the centre to the pole of the zonal harmonics, and let it cut the inner shell at q. Then, by the ordinary surface condition, viz.

$$\frac{dV}{d\nu} + \frac{dV'}{d\nu'} + 4\pi\sigma = 0,$$

we get, if ${}_q\sigma_i$ be the density at q,

$$4\pi {}_q\sigma_i = (2i+1)\frac{\lfloor i \, M_i}{a^i} Y_{ij};$$

therefore $$(2i+1)\iint\sigma_i Q_i dS = 4\pi_q\sigma_i.$$

Now let there be any infinitely thin distribution whatever in the place of the inner shell. We may suppose that it consists of a series of compound shells of degrees 0, 1, 2, &c. Let the density at any point be therefore given by

$$\sigma = \sigma_0 + \sigma_1 + \ldots + \sigma_i + \ldots.$$

Then since $\iint\sigma_i Q_j dS$ vanishes, except when $i=j$, it is obvious we may put, in place of the last written equation,

$$(2i+1)\iint\sigma Q_i dS = 4\pi\,{}_q\sigma_i.$$

Hence $$4\pi\sigma = \iint\sigma dS + \ldots + (2i+1)\iint\sigma Q_i dS + \ldots,$$

which is Laplace's expansion.

§ 7. The theorem of § 2 admits of a physical interpretation. Let a sphere of radius less than OQ be described with Q as centre, and let matter be distributed over it so as to produce at all points of it the same potential as a solid homogeneous globe of unit mass with its centre at O. The potential of the solid globe is $\frac{1}{r}$, and of the distribution

$$\frac{1}{a} - \frac{r'}{a^2}Q_1 + \ldots + (-1)^i\frac{r'^i}{a^{i+1}}Q_i + \ldots,$$

and it is obvious that the distribution consists of a series of compound shells. By performing the operation $(-1)\left[\frac{d}{dh}\right]^i$ on these two expressions, we arrive in one case at the potential due to a compound shell of degree i with its centre at O, and in the other at a complicated distribution on the sphere whose centre is Q, producing at all points inside of it the same potential as the O-shell, viz. equal to

$$\frac{1}{a^{i+1}}\left[\frac{d}{dh}\right]^i r^i Q_i + \&c,$$

at the point Q only the first of these terms exists. It will be seen that this manner of looking at the subject is practically the same thing as finding the distribution of electricity induced in a spherical conductor due to an electrified point outside of it, and suggests an extension of the method to the general case of a spherical conductor placed in a field of electricity.

Let V be the potential due to the field at the centre of

the conductor. Then, by Taylor's theorem, the potential due to the distribution inside of it must be

$$-e^{x\frac{d}{dx}+y\frac{d}{dy}+z\frac{d}{dz}}V,$$

where the centre is taken for origin, and the differential coefficients of V belong to the centre. This may be written

$$-e^{\rho\frac{d}{d\rho}}V,$$

where, after differentiation, we are to understand the differential coefficients as belonging to the centre.

The outside potential due to the distribution is

$$= -\frac{c}{\rho}e^{\frac{c^2}{\rho}\frac{d}{d\rho}}V.$$

Hence the density is given by

$$4\pi\sigma = -\left(\frac{1}{c}+2\frac{d}{d\rho}\right)e^{c\frac{d}{d\rho}}V.$$

In the common case of a point of unit electricity placed at distance f from the centre,

$$\begin{aligned}4\pi\sigma &= -\left(\frac{1}{c}+2\frac{d}{d\rho}\right)e^{c\frac{d}{d\rho}}\frac{1}{f}\\ &= -\left(\frac{1}{c}+2\frac{d}{d\rho}\right)\frac{1}{r}\\ &= -\frac{f^2-c^2}{cr^3},\end{aligned}$$

where r is the distance from the electrified point to the point on the surface under consideration.

§ 8. It may be remarked that the method now given of expanding the potential furnishes a very easy proof of the theorem due to Gauss, that the average potential over any sphere not cutting through any attracting matter is equal to the potential at its centre. The average potential is

$$\frac{1}{4\pi c^2}\iint e^{x\frac{d}{dx}+y\frac{d}{dy}+z\frac{d}{dz}}V\,dS.$$

If we expand the exponential, we have a series of terms of the form

$$\frac{1}{\lfloor i}\left(x\frac{d}{dx}+y\frac{d}{dy}+z\frac{d}{dz}\right)^i V.$$

Now this is a solid harmonic, since it satisfies Laplace's equation; we may therefore write it

$$A_i Y_i r^i.$$

The average potential is therefore

$$\frac{1}{4\pi c^2}\{V\iint dS+A_1 c\iint Y_1 dS+\ldots\}.$$

All the terms are zero except the first, and the result is V.

(To be continued.)

TRANSACTIONS OF SOCIETIES.

London Mathematical Society.

Thursday, February 8th, 1877.—C. W. Merrifield, Esq., F.R.S., *Vice-President*, in the chair. Mr. G. W. Von Tunzelmann was admitted into the Society. The following communications were made. (1) "On the area of the quadrangle formed by the four points of intersection of two conics," Mr. C. Leudesdorf, B.A., Pembroke College, Oxford; (2) "On the numerical value of a certain series," Mr. J. W. L. Glaisher, F.R.S.; (3) "On the general differential equation $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0$, where X, Y are the same quartic functions of x, y respectively," Professor Cayley, F.R.S.; (4) "On the classification of loci and on a theorem in residuation," Professor Clifford, F.R.S.; (1) The four points of intersection are all real, so that the discriminant of the cubic $K^3\Delta+K^2\Theta+K\Theta'+\Delta'=0$ is positive; also it is supposed that the quadrangle formed by the points is non-reentrant. The condition for this may be found from the consideration that no real ellipse or parabola can be drawn round a reentrant quadrangle, and is $\nu^2-4CC'>0$ (where, with the usual notation, $C=ab-h^2$, $C'=a'b'-h'^2$, $\nu=ab'+ba'-2hh'$). The author first investigates the simpler case where the conics both represent pairs of right lines and then takes the general case. (2) The series considered is the following $1-\frac{1}{9}+\frac{1}{25}-\frac{1}{49}+\frac{1}{81}-$&c. The series $\frac{1}{1^n}-\frac{1}{3^n}+\frac{1}{5^n}-\frac{1}{7^n}+$&c., if n be uneven, admits of finite expression as a series of terms involving the first n Bernoullian numbers and having π^n as a factor, but when n is even there is no such formula. The most troublesome case to calculate directly is that of $n=2$, since for this value of n the series converges very slowly; and the calculation is not very easy even when recourse is had to Euler's formula

$$\Sigma u_x=\text{constant}+\int u_x dx-\tfrac{1}{2}u_x+\frac{B_1}{1.2}\frac{du_x}{dx}-\frac{B_2}{1.2.3.4}\frac{d^3u_x}{dx^3}+\text{\&c.}$$

Mr. Glaisher obtains the value of the series to 20 places of decimals. Use is made of a paper communicated by the author at the January meeting of the Society. (3) In this paper Professor Cayley investigated the connexion between Euler's solution of the equation $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0$ and that given by Abel's theorem.

Thursday, March 8th, 1877.—C. W. Merrifield, Esq, F.R.S., *Vice-President*, in the chair. Mr. R. F. Davis, B.A., was admitted into the Society, and Mr. Charles Pendlebury, B.A., was proposed for election.

The following communications were made: "On a new view of the Pascal form," Mr. T. Cotterill, M.A. (the paper turned on dividing the 45 Pascal points into triads; 1°. 15 self-conjugate triads, 2°. 15 diagonal triangles, 3°. 60 triangles, corresponding intersections of two inscribed triangles; 4°. 60 Pascal lines, each Pascal line corresponding to a triangle of 3°. The Pascal points were denoted by Greek letters, thus (α, α', α'') a conjugate triad, so that only 15 Greek letters had to be used. Forming triangles of those points, the author easily obtained the Steiner and Kirkman points as well as other points and properties. "On a class of integers expressible as the sum of two integral squares," Mr. T. Muir, M.A. (the class of integers considered included those whose square root when expressed as a continued fraction has two middle terms in the cycle of partial denominators. A general expression was given for all such integers, and an equivalent expression in the form of the sums of two squares). "Some properties of the double theta functions," Prof. Cayley, F.R.S., (the investigation was founded on papers by Goepel and Rosenhain).

Thursday, April 12th, 1877.—Lord Rayleigh, F.R.S., *President*, in the chair. Mr. Charles Pendlebury, B.A., was elected a member. The following communications were made: "On Hesse's ternary operator and applications," Mr. J. J. Walker, M.A.; "Geometrical illustration of a theorem relating to an irrational function of an imaginary variable, and on the general differential equation $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0$, where X, Y, are the same quartic functions of x, y respectively," Prof. Cayley, F.R.S.; (Profs. H. J. S. Smith and Henrici took part in a discussion on these papers). Mr. C. W. Merrifield, F.R.S., *Vice-President*, having taken the chair, Mr. Harry Hart, M.A., deduced some cases of parallel motion from the consideration that the contra-parallelogram represents the motion of two equal ellipses rolling upon each other, and that of these (*i.e.* parallel motions) two especially were very simple, inasmuch as the motion was obtained in either case by the use of five bars only, and was, moreover, perfectly continuous. Mr. Tucker read an abstract of a paper by Prof. H. W. Lloyd Tanner, on a method of solving partial differential equations which have a general first integral, applied to equations of the third order with two independent variables. The following results were stated to have been obtained: When an equation of the third order admits of a general first integral of the assumed form, the latter is determined by a system of ten linear homogeneous equations of the first order; which are however equivalent to three, and only three, independent equations. The coefficients of this auxiliary system are of two kinds. Those of the first group can be expressed directly in terms of the coefficients of the given equation. Those of the second group are derived indirectly from the coefficients of the given equation; and most simply by expressing that of the ten auxiliary equations only three are mutually independent. The coefficients of the second group may have one, two, or three sets of values, to which correspond one two or three auxiliary systems, each of which serves to determine a first integral. It will be found that in some cases these auxiliary systems are sufficient to determine not only the first, but also the second and even the third and final integrals of the given equation; analogous results are obtained when the method is applied to the equation of the n-th order with m independent variables. The rest of the paper is devoted to a classification of equations of the third order, and to the solution of some examples. It may be noticed that in these examples the author succeeds in solving equations of the second order which involve an arbitary function in their expression. Also some theorems are indicated which may serve as the basis of a general method of solving equations of the second order analogous to those employed in the case of equations of the first order.

R. Tucker, M.A., *Hon. Sec.*

ON THE IRREGULAR FLIGHT OF A TENNIS-BALL.

By *Lord Rayleigh, F.R.S.*

It is well known to tennis players that a rapidly rotating ball in moving through the air will often deviate considerably from the vertical plane. There is no difficulty in so projecting a ball against a vertical wall, that after rebounding obliquely it shall come back in the air and strike the same wall again. It is sometimes supposed that this phenomena is to be explained as a sort of frictional rolling of the rotating ball on the air condensed in front of it, but the actual deviation is in the opposite direction to that which this explanation supposes. A ball projected horizontally and rotating about a vertical axis, deviates from the vertical plane, as if it were rolling on the air *behind* it. The true explanation was given in general terms many years ago by Prof. Magnus, in a paper "On the Deviation of Projectiles," published in the *Memoirs of the Berlin Academy*, 1852, and translated in Taylor's *Scientific Memoirs*, 1853, p. 210. Instead of supposing the ball to remove through air, which at a sufficient distance remains undisturbed, it is rather more convenient to transfer the motion to the air, so that a uniform stream impinges on a ball whose centre maintains its position in space, a change not affecting the relative motion on which alone the mutual forces can depend. Under these circumstances, if there be no rotation, the action of the stream, whether there be friction or not, can only give rise to a force in the direction of the stream, having no lateral component. But if the ball rotate, the friction between the solid surface and the adjacent air will generate a sort of whirlpool of rotating air, whose effect may be to modify the force due to the stream. If the rotation take place about an axis perpendicular to the stream, the superposition of the two motions gives rise on the one side to an augmented, and on the other to a diminished velocity, and consequently to a lateral force urging the ball towards that side on which the motions conspire.

The only weak place in this argument is in the last step, in which it is assumed that the pressure is greatest on the side where the velocity is least. The law that a diminished pressure accompanies an increased velocity is only generally true, on the assumption that the fluid is frictionless and unacted on by external forces; whereas, in the present case, friction is the immediate cause of the whirlpool motion. The actual mode of generation of the lateral force will be perhaps better understood, if we suppose small vertical blades to project from the surface of the ball. On that side of the ball where the motion of the blades is up stream, their anterior faces are in part exposed to the pressure due to the augmented relative velocity, which pressure necessarily operates also on the contiguous spherical surface of the ball. On the other side the relative motion, and therefore also the lateral pressure is less; and thus an uncompensated lateral force remains over.

The principal object of the present note is to propose and solve a problem which has sufficient relation to practice to be of interest, while its mathematical conditions are simple enough to allow of an exact solution being obtained. For this purpose I take the case of a cylinder round which a perfect fluid circulates without molecular rotation. At a great distance from the cylinder the fluid is supposed to move with uniform velocity, and the whole motion is in two dimensions. On these suppositions the stream function, on which the whole motion depends, is of the form

$$\psi = \alpha\left(1 - \frac{a^2}{r^2}\right) r \sin\theta + \beta \log r,$$

where r, θ are the polar coordinates of any point of the fluid, measured from the centre of the cylinder, and the direction of the stream, as pole and initial line respectively, a is the radius of the cylinder, and α, β are constant coefficients proportional respectively to the velocity of the general current and the velocity of circulation round the cylinder. When $r = a$, ψ is constant, shewing that the surface of the cylinder is a stream-line. The radial velocity at any point is given by

$$\frac{d\psi}{r d\theta} = \alpha\left(1 - \frac{a^2}{r^2}\right) \cos\theta;$$

so that, when $r = \infty$ and $\theta = 0$, the radial velocity is α, which is therefore the general velocity of the stream.

At the surface of the cylinder there is no radial velocity, and the magnitude of the tangential velocity is given by

$$\frac{d\psi}{dr} = 2\alpha \sin\theta + \frac{\beta}{a}.$$

Hence, if p_0 be the pressure at a distance, and p the pressure at any point on the surface,

$$2(p - p_0) = \alpha^2 - \left(2\alpha \sin\theta + \frac{\beta}{a}\right)^2,$$

the density of the fluid being taken as unity. Thus the lateral force

$$= \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} (p - p_0)\, a \sin\theta \, d\theta = -\pi\alpha\beta;$$

it is therefore proportional both to the velocity of the motion of circulation, and also to the velocity with which the cylinder moves relatively to the fluid at a distance.

If the velocity of circulation depending on β be small, the character of the stream lines differs but little from that given by

$$\psi = \alpha\left(1 - \frac{a^2}{r^2}\right) r \sin\theta,$$

corresponding to a simple stream; but when β attains a certain point of magnitude, the stream-lines in the neigbourhood of the cylinder become re-entrant.

Sir. W. Thomson has proved that, if in an infinite mass of otherwise quiescent fluid there exist irrotational circulation round a moveable cylinder, the amount of the circulation cannot be changed by any forces applied to the cylinder.* Hence, if the cylinder receive an impulse, it will afterwards move in a circle, and the direction of revolution will be opposite to that of the circulation of the fluid.

It must not be forgotten that the motion of an actual fluid would differ materially from that supposed in the preceding calculation in consequence of the unwillingness of stream-lines to close in at the stern of an obstacle, but this circumstance would have more bearing on the force in the direction of motion than on the lateral component.

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* Vortex Motion, *Edinburgh Transactions*, 1868.

NOTE ON A SYSTEM OF ALGEBRAICAL EQUATIONS.

By Professor *Cayley*.

ASSUME

$$\begin{aligned} x + y + z &= P, \\ yz + zx + xy &= Q, \\ xyz &= R, \end{aligned}$$

$$\begin{aligned} A &= x\,(nyz + Q) - w^2\,(mx + P), \\ B &= y\,(nzx + Q) - w^2\,(my + P), \\ C &= z\,(nxy + Q) - w^2\,(mz + P), \\ \Theta &= -mnR + PQ. \end{aligned}$$

Then $(mz + P)\,B - (my + P)\,C$

$$\begin{aligned} &= (myz + Py)\,(nzx + Q) - (myz + Pz)\,(nxy + Q) \\ &= myz\,(nzx + Q - nxy - Q) + Pnxyz + PQy \\ &\qquad\qquad\qquad\qquad - Pnxyz - PQz \\ &= mnxyz\,(z - y) - PQ\,(z - y) \\ &= (z - y)\,\{mnxyz - PQ\} = (y - z)\,\Theta, \end{aligned}$$

whence, identically

$$\begin{aligned} (mz + P)\,B - (my + P)\,C &= (y - z)\,\Theta, \\ (mx + P)\,C - (mz + P)\,A &= (z - x)\,\Theta, \\ (my + P)\,A - (mx + P)\,B &= (x - y)\,\Theta. \end{aligned}$$

Hence any two of the equations $A = 0$, $B = 0$, $C = 0$ imply the third equation.

We have

$$\begin{aligned} A &= x\,\{(n + 1)\,yz + zx + xy) - w^2\,\{(m + 1)\,x + (y + z)\} \\ &= (x^2 - w^2)\,(y + z) - x\,[(m + 1)\,w^2 - (n + 1)\,yz], \end{aligned}$$

and similarly for B and C. The three equations therefore are

$$\frac{x}{x^2 - w^2} = \frac{y + z}{(m+1)\,w^2 - (n+1)\,yz},$$

$$\frac{y}{y^2 - w^2} = \frac{z + x}{(m+1)\,w^2 - (n+1)\,zx},$$

$$\frac{z}{z^2 - w^2} = \frac{x + y}{(m+1)\,w^2 - (n+1)\,xy},$$

and any two of these equations imply the third equation.

ON THE OCCURRENCE OF THE HIGHER TRANSCENDENTS IN CERTAIN MECHANICAL PROBLEMS.

By *W. H. L. Russell, F.R.S.*

PROFESSOR Schellbach, in his *Lehre von den Elliptischen Integralen*, p. 315, has considered the following problem: "To determine the motion of a particle on the surface of an ellipsoid, subject to the influence of a force varying as the distance, situated in the centre of the ellipsoid." He has shown that the solution depends upon an hyperelliptic differential equation. In endeavouring to extend this theorem I found that mechanical problems might in a great number of cases be resolved by means of hyperelliptic functions and the higher transcendents. I propose to give the results of my investigations in the present paper. These Integral Transcendents have, as is well known, been the subject of investigation by Göpel, Rosenhain, Weierstrass, Riemann, Clebsch, and many other mathematicians. But their researches give rather certain properties of the integrals than the integrals themselves, and do not, as it seems to me, afford methods by which the actual values of the integrals might be found. I have, therefore, endeavoured to find a simple process for their evaluations which I hope to communicate at the end of these contributions.

(1) A tube of small bore in the form of a circle revolves about one of its diameters as a fixed vertical axis, and contains a smooth particle, to determine the motion.

Let the centre of the circle be the origin of coordinates, (a) the radius of the circle, x, y, z the coordinates of the particle where the fixed vertical axis is the axis of (z), θ the angle which the circular tube makes with the plane of (xz), ϕ the angle which the radius vector of particle makes with the axis of (z), P and R reactions on particle in the plane of the tube, and perpendicular to that plane; then, if M and n are masses of tube and particle,

$$\frac{d^2\theta}{dt^2} = -\frac{Ra\sin\phi}{Mk^2},$$

$$m\frac{d^2x}{dt^2} = -R\sin\theta - P\sin\phi\cos\theta,$$

$$m\frac{d^2y}{dt^2} = R\cos\theta - P\sin\phi\sin\theta,$$

$$m\frac{d^2z}{dt^2} = -P\cos\phi - gm,$$

also $\quad x = a\sin\phi\cos\theta,\ y = a\sin\phi\sin\theta,\ z = a\cos\phi;$

wherefore after an elimination which presents no difficulty, we have

$$t = a\sqrt{(m)}\int d\phi\left\{\frac{Mk^2 + ma^2\sin^2\phi}{(c - 2agm\cos\phi)(Mk^2 + ma^2\sin^2\phi) - c_1^2}\right\}^{\frac{1}{2}},$$

where c and c_1 are two constants introduced by the integration, the result is an hyperelliptic function, and may be reduced to the algebraical form by putting $\cos\phi = u$.

This question was proposed many years ago in the Senate-House. I observe that the number of unknown pressures, and therefore of geometrical equations, are less by two than the number of geometrical variables. The same is true of Schellbach's problem just mentioned. In the great majority of cases we have to consider the number of geometrical equations is less by one than the number of geometrical variables.

(2) Two particles P and P' whose masses are (m) and (m') are connected by a straight rod, and compelled to move

in two grooves AP, AP' in one vertical plane to determine the motion.

Let A be the origin of coordinates, the arcs horizontal and vertical at A, x, y coordinates of P, x', y' (where for convenience x' is essentially negative) of P', θ inclination of rod to the horizon, (a) its length, P, P' reactions of grooves, T tension of rod. Then if AP, AP' are inclined to the horizon at angles γ and β, the equations of motion are

$$m\frac{d^2x}{dt^2} = -P\sin\gamma + T\cos\theta,$$

$$m\frac{d^2y}{dt^2} = P\cos\gamma + T\sin\theta - gm,$$

$$m'\frac{d^2x'}{dt^2} = -P'\sin\beta + T\cos\theta,$$

$$m'\frac{d^2y'}{dt^2} = P'\cos\beta - T\sin\theta - gm,$$

$$x + x' = a\cos\theta,\ y - y' = a\sin\theta,\ y = x\tan\gamma,\ y' = x'\tan\beta,$$

we immediately find from these equations

$$t = \int d\theta\,\frac{\sqrt{(A\sin^2\theta + B\sin\theta\cos\theta + C\cos^2\theta)}}{\sqrt{(A'\sin\theta + B'\cos\theta + C')}},$$

where
$$A = \frac{ma^2\sin^2\beta + m'a^2\sin^2\gamma}{\cos^2\beta\cos^2\gamma},$$

$$B = -\frac{2ma^2\tan\beta}{\cos^2\gamma} + \frac{2m'a^2\tan\gamma}{\cos^2\beta},$$

$$C = \frac{ma^2}{\cos^2\gamma} + \frac{m'a^2}{\cos^2\beta},$$

$$A' = -2ga(m\tan\gamma - m'\tan\beta)(\tan\gamma + \tan\beta),$$

$$B' = -2ga(m + m')\tan\beta\tan\gamma(\tan\gamma + \tan\beta),$$

and C' is to be determined by the conditions of motion. This integral may of course be changed into an hyperelliptic function by putting $u = \tan\frac{1}{2}\theta$.

(3) A sphere rolls inside a hollow rough cylinder with its axis horizontal. The centre of gravity of the sphere does not coincide with the centre of the sphere, but these two

centres are always in one vertical plane. To determine the motion.

Let (a) be the radius of the cylinder, (r) the radius of the sphere, c the distance between centre of sphere and centre of gravity of the sphere, let the intersection of the axis of cylinder with given vertical plane be the origin, the coordinate axes, horizontal and vertical with the axis of (y) downwards, ϕ the angle with the line joining the centres of sphere and cylinder makes with the vertical, θ the angle which any radius of sphere makes with its initial position, R the re-action between the two surfaces, and F the friction; then we have from the known properties of roulettes the equation $\theta = \frac{a-r}{r}\phi$. Then the equations of motion are

$$m\frac{d^2x}{dt^2} = -F\cos\phi - R\sin\phi,$$

$$m\frac{d^2y}{dt^2} = F\sin\phi - R\cos\phi + mg,$$

$$mk^2\frac{d^2\theta}{dt^2} = F\{r - c\cos(\theta+\phi)\} - Rc\sin(\theta+\phi),$$

$$x = (a-r)\sin\phi - c\sin\theta,$$

$$y = (a-r)\cos\phi + c\cos\theta,$$

we shall suppose (a) a multiple of (r) and $= nr$, then eliminating, we find

$$t = \frac{n-1}{\sqrt{(2g)}}\int d\phi\left\{\frac{c^2+k^2+r^2-2cr\cos n\phi}{c_1+(n-1)r\cos\phi+c\cos(n-1)\phi}\right\}^{\frac{1}{2}},$$

where the integral may be easily reduced to an hyperelliptic function. This is in fact Euler's Pendulum, the length of the isochronous simple pendulum will be found in Walton.

(To be continued.)

ON CERTAIN SERIES IN TRIGONOMETRY.

By *H. M. Taylor, M.A.*

THE following method of establishing some formulæ in Trigonometry is perhaps worthy of remark. It can easily be shewn that

$$\cos x\pi \cos a\pi - \sin x\pi \sin a\pi = \cos(x+a)\pi$$

$$= \cos a\pi \left(1 - \frac{2x}{1-2a}\right)\left(1 - \frac{2x}{3-2a}\right)\left(1 - \frac{2x}{5-2a}\right) \&c.$$

$$\times \left(1 + \frac{2x}{1+2a}\right)\left(1 + \frac{2x}{3+2a}\right)\left(1 + \frac{2x}{5+2a}\right) \&c. \ldots\ldots\ldots\ldots (1).$$

If we equate the coefficients of x in the two sides of this equation, we obtain

$$-\pi \sin a\pi = \cos a\pi \left(-\frac{2}{1-2a} - \frac{2}{3+2a} - \&c.\right.$$

$$\left. + \frac{2}{1+2a} + \frac{2}{3+2a} + \&c.\right),$$

or $$\tfrac{1}{2}\pi \tan a\pi = \frac{1}{1-2a} - \frac{1}{1+2a} + \frac{1}{3-2a} - \frac{1}{3+2a} + \&c.$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

or $$\frac{\pi}{8a} \tan a\pi = \frac{1}{1-4a^2} + \frac{1}{9-4a^2} + \frac{1}{25-4a^2} + \&c.,$$

which may be written

$$\frac{\pi}{4a} \tan \tfrac{1}{2}a\pi = \frac{1}{1-a^2} + \frac{1}{3^2-a^2} + \frac{1}{5^2-a^2} + \&c. \ldots. (3).$$

Also from equation (1), by equating coefficients of x^2, we obtain $-\frac{1}{2}\pi^2 \cos a\pi$

$= 4 \cos a\pi \times$ product of terms in (2) taken two and two together; whence

$$\frac{1}{(1-2a)^2} + \frac{1}{(1+2a)^2} + \frac{1}{(3-2a)^2} + \frac{1}{(3+2a)^2} + \&c.$$

$$= \frac{\pi^2}{4} \tan^2 a\pi + \frac{\pi^2}{4} = \frac{\pi^2}{4} \sec^2 a\pi,$$

or $$\frac{\pi^2}{4}\sec^2\frac{a\pi}{2} = \frac{1}{(1-a)^2} + \frac{1}{(1+a)^2} + \frac{1}{(3-a)^2} + \frac{1}{(3+a)^2} + \&c. \quad \ldots\ldots\ldots\ldots (4).$$

Similarly, from the equation

$$\sin x\pi \cos a\pi + \cos x\pi \sin a\pi = \sin (x+a)\pi$$

$$= \sin a\pi \left(1 - \frac{x}{1-a}\right)\left(1 - \frac{x}{2-a}\right)\left(1 - \frac{x}{3-a}\right) \&c.$$

$$\times \left(1 + \frac{x}{a}\right)\left(1 + \frac{x}{1+a}\right)\left(1 + \frac{x}{2+a}\right) \&c. \quad \ldots\ldots\ldots\ldots (5),$$

by equating coefficients of x, we obtain

$$\pi \cos a\pi = \sin a\pi \left(\frac{1}{a} + \frac{1}{1+a} + \frac{1}{2+a} + \&c. - \frac{1}{1-a} - \frac{1}{2-a} - \frac{1}{3-a} - \&c.\right),$$

or $$\pi \cot a\pi = \frac{1}{a} - \frac{1}{1-a} + \frac{1}{1+a} - \frac{1}{2-a} + \frac{1}{2+a} + \&c. \ldots (6),$$

or $$= \frac{1}{a} - 2a\left(\frac{1}{1-a^2} + \frac{1}{2^2-a^2} + \frac{1}{3^2-a^2} + \&c.\right),$$

or $$\frac{1 - a\pi \cot a\pi}{2a^2} = \frac{1}{1-a^2} + \frac{1}{2^2-a^2} + \frac{1}{3^2-a^2} + \&c. \ldots (7).$$

Also by equating the coefficients of x^2 in equation (5), we obtain $\frac{1}{2}\pi^2 \sin a\pi$

$= \sin a\pi \times$ product of terms in (6) taken two and two together;

whence

$$\frac{1}{a^2} + \frac{1}{(1-a)^2} + \frac{1}{(1+a)^2} + \frac{1}{(2-a)^2} + \frac{1}{(2+a)^2} + \&c.$$

$$= \pi^2 \cot^2 a\pi + \pi^2 = \pi^2 \operatorname{cosec}^2 a\pi \quad \ldots\ldots\ldots\ldots (8).$$

REDUCTION OF SOME INTEGRALS TO ELLIPTIC FORMS.

By *Artemas Martin.*

(Concluded from Vol. VI., p. 29).

11. PUT

$$I_{11} = \int (a^2 - x^2)^{\frac{1}{2}} (b^2 - x^2)^{\frac{1}{2}} dx$$

$$= b^2 \int \frac{(a^2 - x^2)^{\frac{1}{2}} dx}{(b^2 - x^2)^{\frac{1}{2}}} - \int \frac{x^2 (a^2 - x^2)^{\frac{1}{2}} dx}{(b^2 - x^2)^{\frac{1}{2}}}$$

$$= b^2 [E] - I_2 = \tfrac{1}{3} A + \tfrac{1}{3} (a^2 + b^2) [E] - \tfrac{1}{3} a^2 (a^2 - b^2) [F].$$

12. Put $\quad I_{12} = \int (x^2 - c^2)^{\frac{1}{2}} (a^2 - x^2)^{\frac{1}{2}} dx.$

Let $a^2 - x^2 = y^2$, then $x = (a^2 - y^2)^{\frac{1}{2}}$, $dx = \dfrac{-ydy}{(a^2 - y^2)^{\frac{1}{2}}}$, and

$$I_{12} = -\int \frac{y^2 (a^2 - c^2 - y^2)^{\frac{1}{2}} dy}{(a^2 - y^2)^{\frac{1}{2}}} = -I_3.$$

13. Put $\quad I_{13} = \displaystyle\int \frac{x^2 dx}{(x^2 - c^2)^{\frac{1}{2}} (a^2 - x^2)^{\frac{1}{2}}}.$

Let $a^2 - x^2 = y^2$, and we have

$$I_{13} = -\int \frac{(a^2 - y^2)^{\frac{1}{2}} dy}{(a^2 - c^2 - y^2)^{\frac{1}{2}}} = -[E].$$

14. Put $\quad I_{14} = \displaystyle\int \frac{dx}{(x^2 - c^2)^{\frac{1}{2}} (a^2 - x^2)^{\frac{1}{2}}}.$

Let $a^2 - x^2 = y^2$, and we have

$$I_{14} = -\int \frac{dy}{(a^2 - y^2)^{\frac{1}{2}} (a^2 - c^2 - y^2)^{\frac{1}{2}}} = -[F].$$

15. Put $$I_{15} = \int \frac{x^4 (a^2 - x^2)^{\frac{1}{2}}\, dx}{(b^2 - x^2)^{\frac{1}{2}}}.$$

Decomposing,

$$I_{15} = b^2 \int \frac{x^2 (a^2 - x^2)^{\frac{1}{2}}\, dx}{(b^2 - x^2)^{\frac{1}{2}}} - \int \frac{x^2 \{a^2b^2 - (a^2 + b^2)\, x^2 + x^4\}\, dx}{(a^2 - x^2)^{\frac{1}{2}} (b^2 - x^2)^{\frac{1}{2}}}$$

$$= b^2 I_2 - \tfrac{1}{5} \int \frac{\{3a^2b^2x^2 - 4(a^2 + b^2)\, x^4 + 5x^6\}\, dx}{(a^2 - x^2)^{\frac{1}{2}} (b^2 - x^2)^{\frac{1}{2}}}$$

$$- \tfrac{1}{5} \int \frac{\{2a^2b^2x^2 - (a^2 + b^2)\, x^4\}\, dx}{(a^2 - x^2)^{\frac{1}{2}} (b^2 - x^2)^{\frac{1}{2}}}$$

$$= b^2 I_2 - \tfrac{1}{5} x^2 A - \tfrac{1}{5} \int \frac{\{2a^2b^2x^2 - (a^2 + b^2)\, x^4\}\, dx}{(a^2 - x^2)^{\frac{1}{2}} (b^2 - x^2)^{\frac{1}{2}}}$$

$$= b^2 I_2 - \tfrac{1}{5} x^2 A - \tfrac{1}{5} (a^2 + b^2) \int \frac{x^2 (a^2 - x^2)\, dx}{(b^2 - x^2)^{\frac{1}{2}}}$$

$$+ \tfrac{1}{5} a^2 (a^2 - b^2) \int \frac{x^2 dx}{(a^2 - x^2)^{\frac{1}{2}} (b^2 - x^2)^{\frac{1}{2}}}$$

$$= \tfrac{1}{5} (4b^2 - a^2)\, I_2 - \tfrac{1}{5} x^2 A + \tfrac{1}{5} a^2 (a^2 - b^2) \{b^2 [F] - I_1\}.$$

16. Put $$I_{16} = \int \frac{(x^2 - b^2)^{\frac{1}{2}}\, dx}{(x^2 - a^2)^{\frac{1}{2}}}.$$

Let $$x = \frac{b (a^2 - y^2)^{\frac{1}{2}}}{(b^2 - y^2)^{\frac{1}{2}}},$$

then $$dx = \frac{b (a^2 - b^2)\, dy}{(a^2 - y^2)^{\frac{1}{2}} (b^2 - y^2)^{\frac{3}{2}}}, \quad \frac{(x^2 - b^2)^{\frac{1}{2}}}{(x^2 - a^2)^{\frac{1}{2}}} = \frac{1}{y},$$

and $$I_{16} = b(a^2 - b^2) \int \frac{dy}{(a^2 - y^2)^{\frac{1}{2}} (b^2 - y^2)^{\frac{3}{2}}} = \frac{1}{b} \int \frac{(a^2b^2 - 2b^2y^2 + y^4)\, dy}{(a^2 - y^2)^{\frac{1}{2}} (b^2 - y^2)^{\frac{3}{2}}}$$

$$- \frac{1}{b} \int \frac{(a^2 - y^2)^{\frac{1}{2}}\, dy}{(b^2 - y^2)^{\frac{1}{2}}} + \frac{a^2 - b^2}{b} \int \frac{dy}{(a^2 - y^2)^{\frac{1}{2}} (b^2 - y^2)^{\frac{1}{2}}}$$

$$= \frac{y (a^2 - y^2)^{\frac{1}{2}}}{b (b^2 - y^2)^{\frac{1}{2}}} - \frac{1}{b} [E] + \frac{a^2 - b^2}{b} [F].$$

17. Put $$I_{17} = \int \frac{dx}{(x^2 - a^2)^{\frac{1}{2}} (x^2 - b^2)^{\frac{1}{2}}}.$$

Let $$x = \frac{b (a^2 - y^2)^{\frac{1}{2}}}{(b^2 - y^2)^{\frac{1}{2}}},$$

and we get $$I_{17} = \int \frac{dy}{(a^2 - y^2)^{\frac{1}{2}} (b^2 - y^2)^{\frac{1}{2}}} = [F].$$

18. Put $I_{18} = \int \frac{x^4 dx}{(c^2 - x^2)(b^2 - x^2)^{\frac{1}{2}} (a^2 - x^2)^{\frac{1}{2}}}$.

Decomposing,

$$I_{18} = \int \frac{(a^2 - x^2)^{\frac{1}{2}} dx}{(b^2 - x^2)^{\frac{1}{2}}} - (a^2 + c^2) \int \frac{dx}{(a^2 - x^2)^{\frac{1}{2}} (b^2 - x^2)^{\frac{1}{2}}}$$
$$+ c^4 \int \frac{dx}{(c^2 - x^2)(b^2 - x^2)^{\frac{1}{2}} (a^2 - x^2)^{\frac{1}{2}}}$$
$$= [E] - (a^2 + c^2)[F] + c^4 [\Pi].$$

19. Put $I_{19} = \int [E] dx$. Integrating by parts,

$$I_{19} = x[E] - \int x d[E] = x[E] - \int \frac{x(a^2 - x^2)^{\frac{1}{2}} dx}{(b^2 - x^2)^{\frac{1}{2}}}$$
$$= x[E] + \tfrac{1}{2}(a^2 - x^2)^{\frac{1}{2}} (b^2 - x^2)^{\frac{1}{2}} + \tfrac{1}{2}(a^2 - b^2) \log\{(a^2 - x^2)^{\frac{1}{2}} + (b^2 - x^2)^{\frac{1}{2}}\}.$$

20. Put $I_{20} = \int 2x [E] dx$. Integrating by parts,

$$I_{20} = x^2 [E] - \int x^2 d[E] = x^2 [E] - \int \frac{x^2 (a^2 - x^2)^{\frac{1}{2}} dx}{(b^2 - x^2)^{\frac{1}{2}}}$$
$$= x^2 [E] - I_9.$$

21. Put $I_{21} = \int [F] dx$. Integrating by parts

$$I_{21} = x[F] - \int x d[F]$$
$$= x[F] - \int \frac{x dx}{(a^2 - x^2)^{\frac{1}{2}} (b^2 - x^2)^{\frac{1}{2}}}$$
$$= x[F] + \log\{(a^2 - x^2)^{\frac{1}{2}} + (b^2 - x^2)^{\frac{1}{2}}\}.$$

I will now conclude this series of papers with a few unreduced examples for the exercise of those who may wish to try their skill upon them.

$$I_{22} = \int \frac{x^2 dx}{(x^2 - a^2)^{\frac{1}{2}} (x^2 - b^2)^{\frac{1}{2}}}, \qquad I_{23} = \int \frac{(x^2 - a^2)^{\frac{1}{2}} dx}{(x^2 - b^2)^{\frac{1}{2}}},$$

$$I_{24} = \int \frac{x^2 (x^2 - b^2)^{\frac{1}{2}} dx}{(x^2 - a^2)^{\frac{1}{2}}}, \qquad I_{25} = \int \frac{x^2 (x^2 - a^2)^{\frac{1}{2}} dx}{(x^2 - b^2)^{\frac{1}{2}}},$$

$$I_{26} = \int (x^2 - a^2)^{\frac{1}{2}} (x^2 - b^2)^{\frac{1}{2}} dx, \qquad I_{27} = \int x^2 (x^2 - a^2)^{\frac{1}{2}} (x^2 - b^2)^{\frac{1}{2}} dx.$$

Erie, Pa., U.S.
January 31, 1877.

AN ILLUSTRATION OF THE THEORY OF THE ϑ-FUNCTIONS.

By Professor CAYLEY.

IF X be a given quartic function of x, and if u, or for convenience a constant multiple αu, be the value of the integral $\int\frac{dx}{\sqrt{(X)}}$ taken from a given inferior limit to the superior limit x; then, conversely, x is expressible as a function of u, viz. it is expressible in terms of ϑ-functions of u, where ϑu, or say $\vartheta(u, \mathfrak{F})$ ($\mathfrak{F}$ a parameter upon which the function depends), is given by definition as the sum of a series of exponentials of u; and it is possible from the assumed equation $\alpha u = \int\frac{dx}{\sqrt{(X)}}$, and the definition of ϑu, to obtain by general theory the actual formulæ for the determination of x as such a function of u.

I propose here to obtain these formulæ, in the case where X is a product of real factors, in a less scientific manner, by connecting the function ϑu (as given by such definition) with Jacobi's function Θ, and by reducing the integral $\int\frac{dx}{\sqrt{(X)}}$ by a linear substitution to the form of an elliptic integral; the object being merely to obtain for the case in question the actual formulæ for the expression of x in terms of ϑ-functions of u.

The definition of ϑu (or, when the parameter is expressed $\vartheta(u, \mathfrak{F})$ is

$$\vartheta u = \Sigma\,(-)^s e^{-\mathfrak{F}s^2+2isu},$$

where s has all positive or negative integer values, zero included, from $-\infty$ to $+\infty$ (that is from $-S$ to $+S$, $S=\infty$); the parameter $\mathfrak{F}$, or (if imaginary) its real part must be positive.

ϑu is an even function $\vartheta(-u) = \vartheta u$. Moreover, it is at once seen that we have

$$\vartheta(u+\pi) = \vartheta u,$$
$$\vartheta(u+i\mathfrak{F}) = -e^{\mathfrak{F}-2iu}\vartheta u,$$

whence also $\quad \vartheta(u+m\pi+ni\mathfrak{F}),$

where m and n are any positive or negative integers, is the product of ϑu into an exponential factor, or say simply that it is a multiple of ϑu.

Writing $u=-\frac{1}{2}i\mathfrak{F}$, we have $\vartheta(-\frac{1}{2}i\mathfrak{F})=\vartheta(\frac{1}{2}i\mathfrak{F})$, that is

$$\vartheta(\tfrac{1}{2}i\mathfrak{F})=0,$$

and therefore also $\vartheta\{m\pi+(n+\frac{1}{2})\,i\mathfrak{F}\}=0$.

The above properties are general, but if $\mathfrak{F}$ be real, then k, K, K', q being as in Jacobi (consequently k being real, positive, and less than 1, and K and K' real and positive), then assuming $\mathfrak{F}=\dfrac{\pi K'}{K}$; or, what is the same thing,

$$q\,(=e^{-\frac{\pi K'}{K}})=e^{-\mathfrak{F}},$$

the function ϑ is given in terms of Jacobi's Θ by the equation $\vartheta u=\Theta\left(\dfrac{2Ku}{\pi}\right)$; or, what is the same thing, $\Theta u=\vartheta\left(\dfrac{\pi u}{2K}\right)$.

We hence at once obtain expressions of the elliptic functions $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$ in terms of ϑ, viz. these are

$$\operatorname{sn} u=\frac{-i}{\sqrt{k}}\; e^{-\frac{\pi}{4K}(K'-2iu)}\,\vartheta\left(\frac{\pi u}{2K}+\tfrac{1}{2}i\mathfrak{F}\right)\div\vartheta\left(\frac{\pi u}{2K}\right),$$

$$\operatorname{cn} u=\sqrt{\left(\frac{k'}{k}\right)}\; e^{-\frac{\pi}{4K}(K'-2iu)}\,\vartheta\left(\frac{\pi u}{2K}+\tfrac{1}{2}\pi+\tfrac{1}{2}i\mathfrak{F}\right)\div\vartheta\left(\frac{\pi u}{2K}\right),$$

$$\operatorname{dn} u=\sqrt{k'}\;\vartheta\left(\frac{\pi u}{2K}+\tfrac{1}{2}\pi\right)\div\vartheta\left(\frac{\pi u}{2K}\right).$$

Consider now the integral

$$\int_a\frac{dx}{\sqrt{\{-.\,x-a.\,x-b.\,x-c.\,x-d\}}},\;=\int_a\frac{dx}{\sqrt{(X)}}\text{ suppose,}$$

where a, b, c, d are taken to be real, and in the order of increasing magnitude, viz. it is assumed that $b-a$, $c-a$, $d-a$, $c-b$, $d-b$, $d-c$ are all positive; x considered as the variable under the integral sign is always real; when it is between a and b or between c and d, X is positive, and we assume that $\sqrt{(X)}$ denotes the positive value of the radical; but if x is between b and c, X is negative, and we assume that the sign of $\sqrt{(X)}$ is taken so that $\dfrac{1}{\sqrt{(X)}}$ is equal to a positive multiple of i, and this being so the integral is taken from the inferior limit a to the superior limit x, which is real.

Take x a linear function of y, such that for

$$x=a,\ b,\ c,\ d$$

$$y=0,\ 1,\ \frac{1}{k^2},\ \infty\ \text{ respectively,}$$

so that, x increasing continuously from a to d, y will increase continuously from 0 to ∞.

We have
$$k^2 = \frac{b-a.d-c}{d-b.c-a},$$
$$y = \frac{b-d}{b-a}\,\frac{x-a}{x-d},$$
$$1-y = \frac{d-a}{b-a}\,\frac{x-b}{x-d},$$
$$1-k^2y = \frac{d-a}{c-a}\,\frac{x-c}{x-d};$$
and, thence,
$$\sqrt{(y.1-y.1-k^2y)} = \frac{d-a}{c-a}\sqrt{\left(\frac{d-b}{c-a}\right)}\cdot\frac{\sqrt{(X)}}{(x-d)^2},$$
where $\sqrt{\left(\frac{d-b}{c-a}\right)}$ is taken to be positive: and the sign of $\sqrt{(X)}$ being fixed as above; then for y between 0 and 1 or $> \frac{1}{k^2}$, $y.1-y.1-k^2y$ will be positive, and $\sqrt{(y.1-y.1-k^2y)}$ will also be positive; but y being between 1 and $\frac{1}{k^2}$, $y.1-y.1-k^2y$ will be negative, and the sign of the radical is such that $\frac{1}{\sqrt{(y.1-y.1-k^2y)}}$ is a positive multiple of i.

We have moreover
$$dy = \frac{d-a}{b-a}(d-b)\frac{dx}{(x-d)^2};$$
and therefore
$$\frac{dy}{\sqrt{(y.1-y.1-k^2y)}} = \sqrt{(d-b.c-a)}\frac{dx}{\sqrt{(X)}},$$
where $\sqrt{(d-b.c-a)}$ is positive; or, say,
$$\int_0 \frac{dy}{\sqrt{(y.1-y.1-k^2y)}} = \sqrt{(d-b.c-a)}\int_a \frac{dx}{\sqrt{(X)}}.$$

Hence, writing $y = z^2 = \operatorname{sn}^2 u$, we have
$$2u = \sqrt{(d-b.c-a)}\int_a \frac{dx}{\sqrt{(X)}},$$
and it is to be further noticed that to
$$x = a,\ b,\ c,\ d,$$

correspond $\qquad \operatorname{sn} u = 0,\ 1,\ \frac{1}{k},\ \infty,$

or we may say

$$u = 0,\quad K,\quad K+iK',\quad 2K+iK'.$$

Writing for shortness

$$\frac{2}{\sqrt{(d-b.c-a)}} = \alpha,$$

we have $$\alpha u = \int_a \frac{dx}{\sqrt{(X)}};$$

and moreover $$\alpha K = \int_a^b \frac{dx}{\sqrt{(X)}};$$

$$\alpha(K+iK') = \int_a^c \frac{dx}{\sqrt{(X)}},$$

$$\alpha(2K+iK') = \int_a^d \frac{dx}{\sqrt{(X)}},$$

or if for a moment we write $\int_0^a \frac{dx}{\sqrt{(X)}} = A$, &c., then these equations are

$$\alpha K = B - A,$$

$$\alpha(K+iK') = C - A,$$

$$\alpha(2K+iK') = D - A.$$

Hence $B+C-2A = D-A$, that is $A-B-C+D=0$, or $B-A=D-C$, that is

$$\int_a^b \frac{dx}{\sqrt{(X)}} = \int_c^d \frac{dx}{\sqrt{(X)}},$$

where observe as before that $x=a$ to $x=b$, or $x=c$ to $x=d$, X is positive, and the radical $\sqrt{(X)}$ is taken to be positive.

We have also

$$\alpha K = B - A = \int_a^b \frac{dx}{\sqrt{(X)}},$$

$$\alpha iK' = C - B = \int_b^c \frac{dx}{\sqrt{(X)}},$$

where, as before, from b to c, X is negative, and the sign of the radical is such that $\frac{1}{\sqrt{(X)}}$ is a positive multiple of i; the last formula may be more conveniently written

$$\alpha K' = \int_b^c \frac{dx}{\sqrt{(-X)}},$$

where, from b to c, $-X$ is positive, and $\sqrt{(-X)}$ is also taken to be positive.

Collecting the results, we have

$$\int_a \frac{dx}{\sqrt{(X)}} = \alpha u, \quad \alpha = \frac{2}{\sqrt{(d-b.c-a)}}, \quad k^2 = \frac{b-a.d-c}{d-b.c-a},$$

and also
$$k'^2 = \frac{d-a.c-b}{d-b.c-a},$$

and then conversely

$$x = \frac{a(d-b) + d(b-a)\,\mathrm{sn}^2 u}{(d-b) + (b-a)\,\mathrm{sn}^2 u};$$

or, what is the same thing,

$$\mathrm{sn}^2 u = \frac{b-d.x-a}{b-a.x-d},$$

$$\mathrm{cn}^2 u = \frac{d-a.x-b}{b-a.x-d},$$

$$\mathrm{dn}^2 u = \frac{d-a.x-c}{b-a.x-d};$$

where, in place of the elliptic functions we are to substitute their ϑ-values; it will be recollected that $\mathfrak{J}$ the parameter of the ϑ-functions has the value

$$\mathfrak{J}\left(=\frac{\pi K'}{K}\right) = \pi \int_b^c \frac{dx}{\sqrt{(-X)}} \div \int_a^b \frac{dx}{\sqrt{(X)}},$$

and, as before,
$$K = \frac{1}{\alpha}\int_a^b \frac{dx}{\sqrt{(X)}}.$$

Hence, finally, α, k, k', K, $\mathfrak{J}$ denoting given functions of a, b, c, d, if as above

$$\int_a \frac{dx}{\sqrt{(X)}} = \alpha u,$$

we have conversely

$$\frac{b-d.x-a}{b-a.x-d} = -\frac{1}{k} e^{-\frac{1}{2}\mathfrak{J}+\frac{i\pi u}{2K}}\,\vartheta^2\left(\frac{\pi u}{2K} + \tfrac{1}{2}i\mathfrak{J}\right) \div \vartheta^2\frac{\pi u}{2K},$$

$$\frac{d-a.x-b}{b-a.x-d} = \frac{k'}{k} e^{-\frac{1}{2}\mathfrak{J}+\frac{i\pi u}{2K}}\,\vartheta^2\left(\frac{\pi u}{2K} + \tfrac{1}{2}\pi + \tfrac{1}{2}i\mathfrak{J}\right) \div \vartheta^2\frac{\pi u}{2K},$$

$$\frac{d-a.x-c}{b-a.x-d} = k'\,\vartheta^2\left(\frac{\pi u}{2K} + \tfrac{1}{2}\pi\right) \div \vartheta^2\frac{\pi u}{2K},$$

which are the formulæ in question.

The problem is to obtain them (and that in the more general case where a, b, c, d have any given imaginary values) directly from the assumed equation

$$\int_a \frac{dx}{\surd(X)} = \alpha u,$$

and from the foregoing definition of the function ϑ.

It may be recalled that the function ϑu is a doubly infinite product

$$\vartheta u = \Pi\Pi\left\{1 - \frac{u}{m\pi + (n+\frac{1}{2})\, i\mathfrak{F}}\right\};$$

m and n positive or negative integers from $-\infty$ to $+\infty$; I purposely omit all further explanations as to limits; or, what is the same thing,

$$\vartheta\frac{\pi u}{2K} = \Pi\Pi\left\{1 - \frac{u}{2mK + (2n+1)\, iK'}\right\};$$

and consequently that, disregarding constant and exponential factors, the foregoing expressions of

$$\frac{b-d.x-a}{b-a.x-d},\ \frac{d-a.x-b}{b-a.x-d},\ \frac{d-a.x-c}{b-a.x-d}$$

are the squares of expressions $\frac{X}{W}$, $\frac{Y}{W}$, $\frac{Z}{W}$, where X, Y, Z, W are respectively of the form

$$u\Pi\Pi\left\{1 + \frac{u}{(m, n)}\right\},\ \Pi\Pi\left\{1 + \frac{u}{(\overline{m}, n)}\right\},$$

$$\Pi\Pi\left\{1 + \frac{u}{(\overline{m}, \overline{n})}\right\},\ \Pi\Pi\left\{1 + \frac{u}{(m, \overline{n})}\right\},$$

where $(m, n) = 2mK + 2niK'$, and the stroke over the m or the n denotes that the $2m$ or the $2n$ (as the case may be) is to be changed into $2m+1$ or $2n+1$. But this is a transformation which has apparently no application to the ϑ-functions of more than one variable.

ON THE POTENTIAL OF AN ELLIPTIC CYLINDER.

By Professor *H. Lamb, M.A.*, late Fellow of Trinity College, Cambridge.

THE potential U of an infinitely long homogeneous elliptic cylinder, of unit density, at any external point P, is given, save as to an infinite additive constant, by the formula

$$U = \pi ab \int_{\lambda}^{\infty} \frac{1 - \dfrac{x^2}{a^2+\lambda} - \dfrac{y^2}{b^2+\lambda}}{\sqrt{(a^2+\lambda \,.\, b^2+\lambda)}}\, d\lambda \text{ ....(1)},$$

where x, y denote the coordinates of P with respect to the principal axes of the section of the cylinder made by a plane through P perpendicular to its length, $2a$, $2b$ the lengths of these axes, and where the lower limit of the integral is the positive root of

$$1 - \frac{x^2}{a^2+\lambda} - \frac{y^2}{a^2+\lambda} = 0.$$

Let us transform (1) by assuming

$$a^2 + \lambda = c^2 \left(\frac{e^\eta + e^{-\eta}}{2}\right)^2,$$

where $c^2 = a^2 - b^2$, so that we have also

$$b^2 + \lambda = c^2 \left(\frac{e^\eta - e^{-\eta}}{2}\right)^2.$$

We thus find

$$\frac{d\lambda}{\sqrt{(a^2+\lambda \,.\, b^2+\lambda)}} = 2d\eta,$$

so that the integral in (1) becomes

$$2\int \left\{1 - \frac{4x^2}{c^2(e^\eta + e^{-\eta})^2} - \frac{4y^2}{c^2(e^\eta - e^{-\eta})^2}\right\} d\eta \text{ ....(2)};$$

or, on reduction,

$$\left[2\eta + \frac{4x^2 e^{-\eta}}{c^2(e^\eta + e^{-\eta})} + \frac{4y^2 e^{-\eta}}{c^2(e^\eta - e^{-\eta})}\right] \text{............(3)},$$

taken between the proper limits. Now let us write

$$x + iy = c \sin(\xi' + i\eta'),$$

or

$$\left.\begin{aligned} x &= c \sin\xi' \frac{e^{\eta'} + e^{-\eta'}}{2} \\ y &= c \cos\xi' \frac{e^{\eta'} - e^{-\eta'}}{2} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

We see at once that the limits of integration in (2) are η' and ∞, so that (3) becomes on reduction and omission of additive constants

$$-2\eta' + e^{-2\eta'} \cos 2\xi'.$$

Hence, if V be the function conjugate to U, we have both U and V given by the formula

$$U + iV = \pi ab \{2i(\xi + i\eta) + e^{2i(\xi + i\eta)}\} \ldots\ldots\ldots(5).$$

The chief advantage of the above transformation is that it enables us to find without much difficulty the forms of the equipotential curves and lines of force due to the cylinder. I had occasion *à propos* of certain hydrodynamical problems* to draw the curves by which the first and second terms of the expression on the right-hand side of (5) are separately represented. The curves U= const, V= const, are found at once by combining these according to Maxwell's well-known method.

A few of the equipotential curves are drawn roughly in fig. 2. One of the ellipses confocal with the section of the cylinder is dotted in for comparison.

It may be remarked that (5) contains implicitly the proof of Maclaurin's theorem on the attraction of ellipsoids, as applied to our particular case. For c is the same for all confocal cylinders; it follows then from (4) and (5) that the values of U for two such cylinders are at any external point proportional to the values of the product ab, that is to the masses of equal lengths of the cylinders.

Adelaide,
March 6, 1877.

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* *Quarterly Journal of Mathematics*, December, 1875.

ON SPECIAL METHODS OF INTERPOLATING.

By *W. D. Niven, M.A.*, Trinity College, Cambridge.

THE simplest formulæ of interpolation are expressed in terms of the values of the function at equi-different values of the independent variable. It may, perhaps, be of advantage to consider what modifications are required in formulæ of this kind to render them serviceable when the values of the function are given for vulues of the independent variable which do not proceed by equal intervals. It is obvious there would be a gain in the methods of approximate calculation if there should be found to be no considerable addition to the practical difficulties in using the formula when so modified.

1. We begin with remarking that, in the well-known formula of Lagrange, viz.

$$u_x = u_a \frac{(x-b)(x-c)\dots(x-k)}{(a-b)(a-c)\dots(a-k)} + \dots + u_k \frac{(x-a)(x-b)(x-c)\dots}{(k-a)(k-b)(k-c)\dots},$$

we should have a result which is true, and, for practical purposes, in some cases almost as simple, if in the coefficients of $u_a, u_b, \dots, u_k$, we write $\phi(x)$, $\phi(a)$, ..., $\phi(k)$ instead of $x, a, \dots, k$ where $\phi(x)$ is any function of x; provided always that none of the quantities $\phi(x)$, $\phi(a)$, &c., becomes infinite within the limits of interpolation.

This suggests an examination of interpolation formulæ, in which the data are the values of the function at successive equal intervals of $\phi(x)$.

2. We shall in the first instance suppose, that the successive intervals of $\phi(x)$ differ by unity, and that we know the values of an unknown function $f(x)$ at each of those intervals. The question is then to find an interpolation formulæ for determining approximately the values of $f(x)$.

Let $\phi(x)$ be put equal to z, and suppose that x is accordingly equal to $\psi(z)$. Then $f(x) = f\psi(z)$, and this, by Taylor's theorem, may be expanded in powers of z, viz.

$$f\psi(0) + z\frac{d}{d0}f\psi(0) + \frac{z^2}{\underline{|2}}\left(\frac{d}{d0}\right)^2 f\psi(0) + \dots .$$

If the values of $f(x)$ when $z = 0, 1, 2, 3, \ldots$ be $u_a, u_b, u_c, u_d, \ldots$ we have

$$f\psi(0) = u_a,$$
$$\Delta f\psi(0) \text{ or } \Delta u_a = u_b - u_a,$$
$$\Delta^2 u_a = u_c - 2u_b + u_a,$$
$$\Delta^3 u_a = u_d - 3v_c + 3u_b - u_a,$$
&c.

The above expansion may then be written

$$u_a + \phi(x).\log(1+\Delta)\,u_a + \frac{\overline{\phi(x)}|^2}{\lfloor 2}.\{\log(1+\Delta)\}^2 u_a + \ldots,$$

the general term being

$$\frac{\phi^r(x)}{\lfloor r}.\{\log(1+\Delta)\}^r u_a,$$

in which $\{\log(1+\Delta)\}^r$ must be expanded in powers of Δ, and any difference of u_a, say $\Delta^m u_a$, is to be found according to the above scheme.

3. We have here two remarks to make. In the first place, the formula of the last article is estimated for differences of unity in $\phi(x)$. If, however, the successive differences of $\phi(x)$ be a constant other than unity, say c, the formula will run thus:

$$u_a + \frac{\phi(x)}{c}\log(1+\Delta)\,u_a + \frac{1}{\lfloor 2}\left\{\frac{\phi(x)}{c}\right\}^2 \{\log(1+\Delta)\}^2 u_a + \ldots.$$

In the next place it is often convenient that, while using the differences above defined, viz. the differences of the values of $f(x)$ at the points corresponding to values of z equal to $0, c, 2c,$ &c., we should have the expansion, not in powers of $\phi(x)$, but in powers of $\phi(x) - nc$ where n is an integer, this in fact amounting to changing the origin of z over n intervals without however changing the origin of x. In those circumstances it is easy to see that the proper expansion is

$$e^{n\frac{d}{d0}} u_a + \left\{\frac{\phi(x)}{c} - n\right\}\frac{d}{d0}\,e^{n\frac{d}{d0}} u_a + \text{\&c.},$$

or, what is the same thing,

$$(1+\Delta)^n u_a + \left\{\frac{\phi(x)}{c} - n\right\}\log(1+\Delta)\,(1+\Delta)^n u_a + \ldots,$$

the general term being

$$\frac{1}{\lfloor r}\left\{\frac{\phi(x)}{c}-n\right\}^r \{\log(1+\Delta)\}^r (1+\Delta)^n u_a.$$

4. One of the immediate uses of interpolation formulæ is to determine approximate values of definite integrals. To show the application of the expansion of § 2 to this use, it will suffice to take only three given values of the function $f(x)$. Geometrically, this is the same as knowing the ordinates u_a, u_b, u_c corresponding to three points A, B, C on the axis of x whose abscissæ are a, b, c. The formula becomes

$$f(x)=u_a+\left(\Delta-\frac{\Delta^2}{2}\right)u_a.\phi(x)+\frac{\Delta^2}{2}u_a\{\phi(x)\}^2.$$

Let us now multiply both sides of this equation by $\frac{d\phi}{dx}dx$, and integrate between the values a and c. The corresponding values of $\phi(x)$ are 0 and 2. We get

$$\int_a^c f(x)\frac{d\phi(x)}{dx}dx=\left(2+2\Delta+\frac{\Delta^2}{3}\right)u_a$$

$$=\frac{u_a+4u_b+u_c}{3}.$$

5. As regards $\phi(x)$, we can determine it as a quadratic function in x, by Lagrange's formula, the values at A, B, C being 0, 1, 2; thus

$$\phi(x)=\frac{(x-c)(x-a)}{(b-a)(b-c)}+2\frac{(x-a)(x-b)}{(c-a)(c-b)}.$$

Now let $f(x)\frac{d\phi(x)}{dx}=F(x)$, and let the values of $F(x)$ at the points A, B, C be v_a, v_b, v_c; then taking the above value of $\phi(x)$, we have

$$u_a\left\{\frac{a-c}{(b-a)(b-c)}+2\frac{a-b}{(c-a)(c-b)}\right\}=v_a,$$

$$u_b\left\{\frac{1}{b-c}+\frac{1}{b-a}+2\frac{b-a}{(c-a)(c-b)}\right\}=v_b,$$

$$u_c\left\{\frac{c-a}{(b-a)(b-c)}+\frac{2}{c-a}+\frac{2}{c-b}\right\}=v_c.$$

If we now substitute these values of u_a, u_b, u_c in terms of v_a, v_b, v_c in the formula of the last article, viz.

$$\int_a^c F(x)\,dx = \frac{u_a + 4u_b + u_c}{3},$$

we shall have an approximate value of the integral.

The same method of treatment may obviously be employed whatever be the number of points A, B, C, ..., the only difficulty being the reduction of the equations connecting the u's and v's.

6. One of the simplest cases of interpolation, other than the common case with the equi-different abscissæ, is when the logarithms of the abscissæ are equi-different, or when the distances OA, OB, OC ... are in geometrical progression. Let there be $(2n+1)$ of these distances, and let ξ be the $(n+1)^{\text{th}}$, then the general expression for $f(x)$ may be taken

$$(1+\Delta)^n u_a + \log\frac{x}{\xi}\,(1+\Delta)^n \log(1+\Delta)\,u_a + \dots$$
$$+ \frac{1}{\underline{|r}}\left(\log\frac{x}{\xi}\right)^r (1+\Delta)^n \{\log(1+\Delta)\}^r u_a + \dots.$$

In this formula the differences are estimated for differences of unity in the logarithm. If the latter differences are not unity, we must divide $\log\frac{x}{\xi}$ wherever it occurs by $\log\frac{OB}{OA}$. In like manner we might establish a formula when the number of distances OA, OB, ... is even.

7. There is a peculiarity about the case of three points which makes it worthy of special discussion. The three points being A, B, C, let $AB = \alpha$ and $BC = \beta$, in which β is greater than α. Then if we choose the origin O, so that $OA = \xi e^{-c}$, $OB = \xi$, $OC = \xi e^{c}$, we shall have

$$c = \log\frac{\beta}{\alpha},$$

and
$$\xi = \frac{\alpha\beta}{\beta - \alpha}.$$

Now making use of the formula of § 6, we get

$$f(x) = u_b + \frac{u_c - u_a}{2}\,\frac{\log\frac{(\beta-\alpha)x}{\alpha\beta}}{\log\frac{\beta}{\alpha}} + \frac{u_c - 2u_b + u_a}{2}\left\{\frac{\log\frac{(\beta-\alpha)x}{\alpha\beta}}{\log\frac{\beta}{\alpha}}\right\}^2.$$

The peculiarity of the case of three given values is, that B may be anywhere between A and C, subject to the condition $\beta > \alpha$, and therefore we are able to select the origin. This is of course in our power for other forms of $\phi(x)$ besides $\log x$, for we have in general as an equation to determine the distance of the origin from B,

$$\phi(\xi - \alpha) + \phi(\xi + \beta) = 2\phi(\xi).$$

In the ordinary system of equidistant abscissæ, this equation makes the value of ξ infinite.

It is to be observed, that the form of the result given above for $f(x)$ is true for the ordinary system of logarithms.

8. Pursuing the method of § 4, we might prove that the value of any definite integral

$$\int_a^c F(x)\,dx,$$

between the limits of interpolation, is approximately

$$\frac{\log\beta - \log\alpha}{\beta - \alpha}\,\frac{\alpha^2 v_a + 4\alpha\beta v_b + \beta^2 v_c}{3}.$$

9. The method of interpolating which has been developed in the preceding articles was used in finding, from Mr. Bashforth's tables for the resistance of shot, an approximate representation of the law of the resistance in terms of a simple power of the velocity. As this seems an interesting application of the method we will now show how it may be made use of for that purpose. We will employ an interpolation in logarithms taking the values for five velocities. The formula applicable will first be stated.

Let the five abscissæ points be A, B, C, D, E so situated that their distances from O are in a geometrical ratio e^c from O. Also let $OC = \xi$. Then

$$\begin{aligned} f(x) = u_c &+ \frac{u_a - 8u_b + 8u_d - u_e}{6}\log\frac{x}{\xi} \\ &+ \frac{-u_a + 16u_b - 30u_c + 16u_d - u_e}{6}\left(\log\frac{x}{\xi}\right)^2 \\ &+ \frac{-2u_a + 4u_b - 4u_d + 2u_e}{3}\left(\log\frac{x}{\xi}\right)^3 \\ &+ \frac{2u_a - 8u_b + 12u_c - 8u_d + 2u_e}{3}\left(\log\frac{x}{\xi}\right)^4. \end{aligned}$$

The five numbers are taken from the tables for ogival-headed shot. The law of retardation is formulated by

$$\frac{d^2}{W} K_v \left(\frac{v}{1000}\right)^3,$$

being the diameter of the shot and W its weight in lbs., and K_v is tabulated for every 10 feet of velocity between certain limits.

Putting
$$\begin{aligned}
a &= 1400 \times e^{-\frac{1}{5}} = 1146{\cdot}1, \\
b &= 1400 \times e^{-\frac{1}{10}} = 1266{\cdot}8, \\
c &= 1400 \qquad\quad\; = 1400, \\
d &= 1400 \times e^{+\frac{1}{10}} = 1547{\cdot}2, \\
e &= 1400 \times e^{+\frac{1}{5}} = 1710.
\end{aligned}$$

We then have, from the tables, as the values of K_v,

$$\begin{aligned}
K_a &= 108{\cdot}1, \\
K_b &= 108{\cdot}5, \\
K_c &= 104, \\
K_d &= 93{\cdot}2, \\
K_e &= 84{\cdot}1.
\end{aligned}$$

Hence
$$K_v = 104{\cdot}0 - 164 \log \frac{v}{1400} + 1416 \left(\log \frac{v}{1400}\right)^2 + 4400 \left(\log \frac{v}{1400}\right)^3 + 63000 \left(\log \frac{v}{1400}\right)^4.$$

For values of v in the vicinity of 1400 the quantity $\log \frac{v}{1400}$ is small; for example, between the limits $1400e^{-\frac{1}{20}}$ and $1400e^{\frac{1}{20}}$, we may consider K_v as being approximately represented by the first two terms. Now if K_v between hose limits be of the form Av^m, we have

$$\begin{aligned}
K_v &= A1400^m e^{m \log \frac{v}{1400}} \\
&= A1400^m \left(1 + m \log \frac{v}{1400} + \&c.\right).
\end{aligned}$$

Hence
$$m = -\tfrac{164}{104}.$$

This shows that the retardation of the shot, when it moves with a velocity 1400 feet per second, varies approximately as $v^{\frac{3}{2}}$.

Nearly the same result is readily got at by the use of the common interpolation formula, the values of K_v for intervals of 50 feet difference of v being the given values.

NOTE ON A THEOREM IN HYDRODYNAMICS.

By Professor *H. Lamb, M A.*

THE well-known theorem of Helmholtz, that in a moving liquid the vortex-lines always consist each of the same series of particles, was deduced by him from certain equations obtained by elimination of p (the pressure) from the ordinary hydrodynamical equations. Thomson afterwards gave a proof, founded on the theorem that the "circulation," that is, with the usual notation, the value of

$$\int(udx + vdy + wdz) \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(1)},$$

taken round any circuit moving with the fluid, is constant with regard to the time. The object of this note is to exhibit the connection between these two modes of proof.

Let A, B, C be the areas of the projections on the coordinate planes of any circuit moving with the fluid, and let $\frac{\delta}{\delta t}$ denote a differentiation following the motion of the fluid. We easily find

$$\frac{\delta A}{\delta t} = \int(vdz - wdy),$$

the integration being taken round the circuit. If the latter be infinitely small, this becomes

$$\frac{\delta A}{\delta t} = A\left(\frac{dv}{dy} + \frac{dw}{dz}\right) - B\frac{dv}{dx} - C\frac{dw}{dx}$$

$$= A\theta - A\frac{du}{dx} - B\frac{dv}{dx} - C\frac{dw}{dx} \quad \ldots\ldots\ldots\ldots \text{(2)},$$

where
$$\theta = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}.$$

Now Thomson has shown that the integral (1) taken round an infinitely small circuit is equal to

$$2(\xi A + \eta B + \zeta C),$$

where ξ, η, ζ are the instantaneous angular velocities

$$\tfrac{1}{2}\left(\frac{dw}{dy} - \frac{dv}{dz}\right), \text{ \&c., \&c.,}$$

so that his circulation-theorem, just referred to, becomes, for the case in question,

$$\frac{\delta}{\delta t}(\xi A + \eta B + \zeta C) = 0,$$

that is, by (2), and similar equations,

$$\begin{aligned} &A\left(\frac{\delta\xi}{\delta t} - \xi\frac{du}{dx} - \eta\frac{du}{dy} - \zeta\frac{du}{dz}\right) \\ &+ B\left(\frac{\delta\eta}{\delta t} - \xi\frac{dv}{dx} - \eta\frac{dv}{dy} - \zeta\frac{dv}{dz}\right) \\ &+ C\left(\frac{\delta\zeta}{\delta t} - \xi\frac{dw}{dx} - \eta\frac{dw}{dy} - \zeta\frac{dw}{dz}\right) \\ &+ \quad (A\xi + B\eta + C\zeta)\,\theta \qquad = 0 \ldots\ldots \text{(3)}. \end{aligned}$$

Since this must hold for *any* infinitely small circuit, the coefficients of A, B, C on the left-hand side of this equation must separately vanish, which gives, with $\theta = 0$, Helmholtz's equations above referred to.

Thomson's investigation applies not merely to a liquid but to any fluid in which the density is a function of p. Mr. Nanson has shown that this extension of Helmholtz's original theorem follows from the equations which are implied in (3) when θ does not vanish.

ON THE PRODUCT $1^1.2^2.3^3...n^n$.

By *J. W. L. Glaisher.*

§1. It is easy to show, by means of the formula

$$\Sigma u_x = \text{const.} + \int u_x dx - \tfrac{1}{2}u_x + \frac{B_1}{1.2}\frac{du_x}{dx} - \frac{B_2}{1.2.3.4}\frac{d^3u_x}{dx^3} + \&\text{c.} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

that

$$1^1.2^2.3^3\ldots n^n = An^{\frac{1}{2}n^2+\frac{1}{2}n+\frac{1}{12}}e^{-\frac{1}{4}n^2}\left(1+\frac{1}{720n^2}-\frac{1433}{7257600n^4}+\&\text{c.}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where A is a constant independent of n.

The numerical value of A is best determined by giving a particular value to n, and for this purpose the form in which the equation is directly given by (1), viz.

$$1\log 1 + 2\log 2 \ldots + n\log n = \log A$$
$$+ \left(\tfrac{1}{2}n^2+\tfrac{1}{2}n+\tfrac{1}{12}\right)\log n - \frac{1}{4}n^2 + \frac{1}{720}\cdot\frac{1}{n^2} - \frac{1}{5040}\cdot\frac{1}{n^4} + \&\text{c.}\ldots(3),$$

is rather preferable to (2).

Putting therefore $n = 10$, and multiplying throughout by M, $= 0.43429\ldots$, to convert the hyperbolic into Briggian logarithms, we have

$$1\log_{10}1 + 2\log_{10}2\ldots + 10\log_{10}10 = \log_{10}A + 50 + 5 + \tfrac{1}{12} - 25M$$
$$+ M\left(\frac{1}{72{,}000} - \frac{1}{50{,}400{,}000}\right.$$
$$\left.+ \frac{1}{10{,}080{,}000{,}000} - \frac{1}{950{,}400{,}000{,}000} + \&\text{c.}\right).$$

The left-hand side $= 44{\cdot}33401\ 00057\ 699\ldots$, and the series $\frac{1}{72{,}000} - \&\text{c.} = 0{\cdot}00001\ 38691\ 457\ldots$, which when multiplied by M becomes $0{\cdot}00000\ 60232\ 9\ldots$, and it is thus found that

$$\log_{10}A = 0{\cdot}10803\ 26967\ 2\ldots,$$

whence $$A = 1{\cdot}28242\ 7130,$$

in which value of A, as it is obtained by ten-figure logarithms, the last figure may be slightly in error.

§ 2. The above method, viz. by giving a particular value to n in (2) or (3), would probably always afford the best practical means of obtaining the numerical value of A to any number of decimal places; but it will be noticed, that since the series in (3) is semi-convergent, none of the equations written above contain a perfect analytical definition of the constant A, as it is given in terms of series that ultimately diverge. The object of what follows in §§ 3 and 4 is to obtain an expression for A in terms of a convergent series.

§ 3. When n is very large, we have from (2)

$$1^1.2^2.3^3\ldots n^n = An^{\frac{1}{2}n^2+\frac{1}{2}n+\frac{1}{12}}e^{-\frac{1}{4}n^2} \quad \ldots\ldots\ldots\ldots\ldots (4).$$

Multiply by $\qquad 2^1.2^2.2^3\ldots 2^n, \; = 2^{\frac{1}{2}n(n+1)},$

and we have

$$2^1.4^2.6^3\ldots(2n)^n = A2^{\frac{1}{2}n^2+\frac{1}{2}n}\,n^{\frac{1}{2}n^2+\frac{1}{2}n+\frac{1}{12}}e^{-\frac{1}{4}n^2},$$

which, when squared, becomes

$$2^2.4^4.6^6\ldots(2n)^{2n} = A^2 2^{n^2+n}\,n^{n^2+n+\frac{1}{6}}e^{-\frac{1}{2}n^2},$$

suppose n even, and write $\frac{1}{2}n$ for n, and this is

$$2^2.4^4.6^6\ldots n^n = A^2 2^{-\frac{1}{6}}\,n^{\frac{1}{4}n^2+\frac{1}{2}n+\frac{1}{6}}e^{-\frac{1}{8}n^2}.$$

Square again and divide by (4), which gives

$$\frac{2^2.4^4.6^6\ldots n^n}{1^1.3^3.5^5\ldots(n-1)^{n-1}} = A^3 2^{-\frac{1}{3}}\,n^{\frac{1}{2}n+\frac{1}{4}} \quad \ldots\ldots\ldots (5).$$

§ 4. Now let

$$u = \left(\frac{1+x}{1}\right)^1\left(\frac{3+x}{3}\right)^3\left(\frac{5+x}{5}\right)^5\ldots\left(\frac{n-1+x}{n-1}\right)^{n-1},$$

whence

$$\begin{aligned}
\log u = x &- \tfrac{1}{2}\,x^2 + \tfrac{1}{3}\,x^3 - \tfrac{1}{4}\,x^4 + \&c.\\
+ x &- \tfrac{1}{2}\,\frac{x^2}{3} + \tfrac{1}{3}\,\frac{x^3}{3^2} - \tfrac{1}{4}\,\frac{x^4}{3^3} + \&c.\\
+ x &- \tfrac{1}{2}\,\frac{x^2}{5} + \tfrac{1}{3}\,\frac{x^3}{5^2} - \tfrac{1}{4}\,\frac{x^4}{5^3}\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
+ x &- \tfrac{1}{2}\,\frac{x^2}{n-1} + \tfrac{1}{3}\,\frac{x^3}{(n-1)^2} - \tfrac{1}{4}\,\frac{x^4}{(n-1)^3} + \&c.\\
= \tfrac{1}{2}nx &- \tfrac{1}{2}\,Qx^2 + \tfrac{1}{3}\,s_2x^3 - \tfrac{1}{4}\,s_3x^4 + \&c.,
\end{aligned}$$

where $$Q = 1 + \tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{7} \ldots + \frac{1}{n-1},$$

and $$s_r = 1 + \frac{1}{3^r} + \frac{1}{5^r} + \frac{1}{7^r} + \&c.$$

Put $x = 1$, and we have

$$\log\left\{\frac{2^1}{1^1}\cdot\frac{4^3}{3^3}\cdot\frac{6^5}{5^5}\cdots\frac{n^{n-1}}{(n-1)^{n-1}}\right\} = \tfrac{1}{2}n - \tfrac{1}{2}Q + \tfrac{1}{3}s_2 - \tfrac{1}{4}s_3 + \&c.,$$

or, as we may write it,

$$\frac{2^1.4^3.6^5\ldots n^{n-1}}{1^1.3^3.5^5\ldots(n-1)^{n-1}} = \exp\left(\tfrac{1}{2}n - \tfrac{1}{2}Q + \tfrac{1}{3}s_2 - \tfrac{1}{4}s_3 + \&c.\right)^*$$

Multiply by $\quad 2.4.6\ldots n, = \pi^{\frac{1}{2}} n^{\frac{1}{2}n+\frac{1}{2}} e^{-\frac{1}{2}n},$

and this becomes

$$\frac{2^2.4^4.6^6\ldots n^n}{1^1.3^3.5^5\ldots(n-1)^{n-1}} = \pi^{\frac{1}{2}} n^{\frac{1}{2}n+\frac{1}{2}} \exp\left(-\tfrac{1}{2}Q + \tfrac{1}{3}s_2 - \tfrac{1}{4}s_3 + \&c\right) \ldots (6),$$

it which it only remains to determine the value of Q. This is easily effected for, γ denoting Euler's constant 0·57721..., we know that

$$1 + \tfrac{1}{2} + \tfrac{1}{3} \ldots + \frac{1}{n} = \gamma + \log n,$$

whence $$\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{6} \ldots + \frac{1}{2n} = \tfrac{1}{2}\gamma + \tfrac{1}{2}\log n,$$

and $$1 + \tfrac{1}{2} + \tfrac{1}{3} \ldots + \frac{1}{2n} = \gamma + \log(2n);$$

therefore, by subtraction,

$$1 + \tfrac{1}{3} + \tfrac{1}{5} \ldots + \frac{1}{2n-1} = \tfrac{1}{2}\gamma + \log 2 + \tfrac{1}{2}\log n,$$

so that $$Q = \tfrac{1}{2}\gamma + \tfrac{1}{2}\log 2 + \tfrac{1}{2}\log n.$$

Substituting this value of Q in (6), that equation becomes

$$\frac{2^2.4^4.6^6\ldots n^n}{1^1.3^3.5^5\ldots(n-1)^{n-1}} = 2^{-\frac{1}{4}}\pi^{\frac{1}{2}} n^{\frac{1}{2}n+\frac{1}{4}} \exp\left(-\tfrac{1}{4}\gamma + \tfrac{1}{3}s_2 - \tfrac{1}{4}s_3 + \&c.\right).$$

Comparing this with (5), we have

$$A^3 = 2^{\frac{1}{12}}\pi^{\frac{1}{2}} \exp\left(-\tfrac{1}{4}\gamma + \tfrac{1}{3}s_2 - \tfrac{1}{4}s_3 + \&c.\right),$$

---

* exp (u) is written for e^u when u is complicated.

and therefore

$$A = 2^{\frac{1}{36}} \pi^{\frac{1}{6}} \exp \tfrac{1}{3} \left(- \tfrac{1}{4}\gamma + \tfrac{1}{3}s_2 - \tfrac{1}{4}s_3 + \tfrac{1}{5}s_4 - \&c\right) \ldots\ldots (7),$$

which is the value of A expressed in terms of a convergent numerical series.

§ 5. To obtain the numerical value of A from this series we notice that the quantity in brackets subject to the exponential sign is

$$- \tfrac{1}{4}\gamma + \tfrac{1}{3}s_2 - \tfrac{1}{4}s_3 + \tfrac{1}{5}s_4 - \&c.,$$

which $= - \tfrac{1}{4}\gamma + (\tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \&c.) + \tfrac{1}{3}(s_2 - 1) - \tfrac{1}{4}(s_3 - 1) + \&c.$

$$= - \tfrac{1}{4}\gamma + \log 2 - \tfrac{1}{2} + \tfrac{1}{3}(s_2 - 1) - \tfrac{1}{4}(s_3 - 1) + \&c,$$

whence

$$A^3 = 2^{\frac{13}{12}} \pi^{\frac{1}{2}} \exp \left\{- \tfrac{1}{4}\gamma - \tfrac{1}{2} + \tfrac{1}{3}(s_2 - 1) - \tfrac{1}{4}(s_3 - 1) + \&c\right\};$$

and, therefore, $3 \log_{10} A$

$$= \tfrac{13}{12} \log_{10} 2 + \tfrac{1}{2} \log_{10} \pi - M \left\{\tfrac{1}{4}\gamma + \tfrac{1}{2} - \tfrac{1}{3}(s_2 - 1) + \tfrac{1}{4}(s_3 - 1) - \&c.\right\}.$$

The series in brackets is found on calculation to be

$$= 0{\cdot}57701\ 09574\ 3\ldots,$$

which, when multiplied by M,

$$= 0{\cdot}25059\ 26748\ 1,$$

and $\qquad \tfrac{13}{12} \log 2 + \tfrac{1}{2} \log \pi = 0{\cdot}57469\ 07649\ 8;$

therefore $\qquad 3 \log_{10} A = 0{\cdot}32409\ 80901\ 7,$

and $\qquad \log_{10} A = 0{\cdot}10803\ 26967\ 2,$

agreeing with the value found in § 1 to the last figure.

§ 6. The series in (7) can readily be expressed as a definite integral, for if

$$S_r = 1 + \frac{1}{2^r} + \frac{1}{3^r} + \frac{1}{4^r} + \&c.,$$

then $\qquad s_r = S_r \left(1 - \dfrac{1}{2^r}\right),$

so that $- \tfrac{1}{4}\gamma + \tfrac{1}{3}s_2 - \tfrac{1}{4}s_3 + \tfrac{1}{5}s_4 - \&c.$

$$= - \tfrac{1}{4}\gamma + \frac{S_2}{3}\left(1 - \frac{1}{2^2}\right) - \frac{S_3}{4}\left(1 - \frac{1}{2^3}\right) + \&c. \ldots\ldots (8).$$

Now $\log \Gamma(1 + x) = - \gamma x + \tfrac{1}{2}S_2 x^2 - \tfrac{1}{3}S_3 x^3 + \tfrac{1}{4}S_4 x^4 - \&c.,$

whence

$$\int_0^x \log\Gamma(1+x)\,dx = -\tfrac{1}{2}\gamma x^2 + \frac{S_2}{2.3}x^3 - \frac{S_3}{3.4}x^4 + \frac{S_4}{4.5}x^5 - \&c.,$$

and $\quad x\log\Gamma(1+x)\,dx = -\gamma x^2 + \tfrac{1}{2}S_2x^3 - \tfrac{1}{3}S_3x^4 + \tfrac{1}{4}S_4x^5 - \&c.$

so that $\quad x\log\Gamma(1+x) - \int_0^x \log\Gamma(1+x)\,dx$

$$= -\tfrac{1}{2}\gamma x^2 + \tfrac{1}{3}S_2x^3 - \tfrac{1}{4}S_3x^4 + \&c.$$

Put $x=1$, and we have

$$-\int_0^1 \log\Gamma(1+x)\,dx = -\tfrac{1}{2}\gamma + \tfrac{1}{3}S_2 - \tfrac{1}{4}S_3 + \&c.;$$

that is $\quad 1 - \tfrac{1}{2}\log(2\pi) = -\tfrac{1}{2}\gamma + \tfrac{1}{3}S_2 - \tfrac{1}{4}S_3 + \&c. \ldots\ldots.(9).$

Put $x=\tfrac{1}{2}$, and we have

$$\tfrac{1}{2}\log\Gamma(\tfrac{3}{2}) - \int_0^{\frac{1}{2}} \log\Gamma(1+x)\,dx$$

$$= -\tfrac{1}{2}\gamma.\frac{1}{2^2} + \tfrac{1}{3}S_2.\frac{1}{2^3} - \tfrac{1}{4}S_3.\frac{1}{2^4} + \&c.;$$

that is, $\quad \log(\tfrac{1}{2}\sqrt{\pi}) - 2\int_0^{\frac{1}{2}} \log\Gamma(1+x)\,dx$

$$= -\tfrac{1}{2}\gamma.\tfrac{1}{2} + \tfrac{1}{3}S_2.\frac{1}{2^2} - \tfrac{1}{4}S_3.\frac{1}{2^3} + \&c.\ldots\ldots\ldots(10),$$

whence from (9) and (10)

$$1 + \tfrac{1}{2}\log 2 - \log\pi + 2\int_0^{\frac{1}{2}} \log\Gamma(1+x)\,dx$$

$$= -\tfrac{1}{4}\gamma + \frac{S_2}{3}\left(1 - \frac{1}{2^2}\right) - \frac{S_3}{4}\left(1 - \frac{1}{2^3}\right) + \&c.$$

$$= -\tfrac{1}{4}\gamma + \tfrac{1}{3}s_2 - \tfrac{1}{4}s_3 + \&c.$$

from (8), thus

$$A^3 = 2^{\frac{7}{12}}\pi^{-\frac{1}{2}}\exp\{1 + 2\textstyle\int_0^{\frac{1}{2}}\log\Gamma(1+x)\,dx\},$$

and therefore

$$A = 2^{\frac{7}{36}}\pi^{-\frac{1}{6}}\exp\{\tfrac{1}{3} + \tfrac{2}{3}\textstyle\int_0^{\frac{1}{2}}\log\Gamma(1+x)\,dx\}.$$

ON THE TRIPLE THETA-FUNCTIONS.

By Professor *Cayley*.

As a specimen of mathematical notation, viz. of the notation which appears to me the easiest to *read* and also to *print*, I give the definition and demonstration of the fundamental properties of the triple theta-functions.

Definition. $\vartheta(U, V, W) = \Sigma \exp \Theta$, where

$$\Theta = (A, B, C, F, G, H)(l, m, n)^2 + 2(U, V, W)(l, m, n),$$

Σ denoting the sum in regard to all positive and negative integer values from $-\infty$ to $+\infty$ (zero included) of l, m, n respectively.

$\vartheta(U, V, W)$ is considered as a function of the arguments (U, V, W), and it depends also on the parameters (A, B, C, F, G, H).

First property. $\vartheta(U, V, W) = 0$, for

$$U = \tfrac{1}{2}\{x\pi i + (A, H, G)(\alpha, \beta, \gamma)\},$$

$$V = \tfrac{1}{2}\{y\pi i + (H, B, F)(\alpha, \beta, \gamma)\},$$

$$W = \tfrac{1}{2}\{x\pi i + (G, F, C)(\alpha, \beta, \gamma)\},$$

$x, y, z, \alpha, \beta, \gamma$ being any positive or negative integer numbers, such that $\alpha x + \beta y + \gamma z =$ odd number.

Demonstration. It is only necessary to show, that to each term of ϑ there corresponds a second term, such that the indices of the two exponentials differ by an odd multiple of πi.

Taking l, m, n as the integers which belong to the one term, those belonging to the other term are

$$-(l+\alpha),\ -(m+\beta),\ -(n+\gamma),$$

(where observe that one at least of the numbers α, β, γ being odd, this system of values is not in any case identical with l, m, n). The two exponents then are

$$\Theta, = (A, B, C, F, G, H)(l, m, n)^2 + 2(U, V, W)(l, m, n),$$

and $$\Theta', = (A, B, C, F, G, H)(l+\alpha, m+\beta, n+\gamma)^2 - 2(U, V, W)(l+\alpha, m+\beta, n+\gamma),$$

viz. the value of Θ' is

$$= (A, B, C, F, G, H)(l, m, n)^2 + (A, B, C, F, G, H)(\alpha, \beta, \gamma)^2$$
$$+ 2(A, B, C, F, G, H)(l, m, n)(\alpha, \beta, \gamma)$$
$$- 2(U, V, W)(l+\alpha, m+\beta, n+\gamma),$$

and we then have

$$\Theta' - \Theta = 2(A, B, C, F, G, H)(l, m, n)(\alpha, \beta, \gamma)$$
$$+ (A, B, C, F, G, H)(\alpha, \beta, \gamma)^2$$
$$- 2(U, V, W)(2l+\alpha, 2m+\beta, 2n+\gamma).$$

Substituting herein for U, V, W their values, the last term is

$$= -\{(2l+\alpha)x + (2m+\beta)y + (2n+\gamma)z\}$$
$$- 2(A, B, C, F, G, H)(l, m, n)(\alpha, \beta, \gamma)$$
$$- (A, B, C, F, G, H)(\alpha, \beta, \gamma)^2,$$

and thence

$$\Theta' - \Theta = - \{(2l+\alpha)x + (2m+\beta)y + (2n+\gamma)z\}\,\pi i,$$

which proves the theorem.

As to the notation, remark that after (A, B, C, F, G, H) has been once written out in full, we may instead of

$(A, B, C, F, G, H)(l, m, n)^2$, &c., write $(A, \ldots)(l, m, n)^2$, &c., and that we may use the like abbreviations

$(A, \ldots)(l, m, n)$, to denote $(A, H, G)(l, m, n)$ respectively,
$(H, \ldots)(l, m, n)$, „ $(H, B, F)(l, m, n)$ „
$(G, \ldots)(l, m, n)$, „ $(G, F, C)(l, m, n)$ „

these are not only abbreviations, but they make the formulæ actually clearer, as bringing them into a smaller compass; and I accordingly use them in the demonstration which follows.

Second Property. If U_1, V_1, W_1 denote

$$U + x\pi i + (A, H, G)(\alpha, \beta, \gamma),$$
$$V + y\pi i + (H, B, F)(\alpha, \beta, \gamma),$$
$$W + z\pi i + (G, F, C)(\alpha, \beta, \gamma),$$

respectively, where x, y, z, α, β, γ are any positive or negative integers (zero values admissible), then

$$\vartheta(U_1, V_1, W_1) = \exp\{-(A, B, C, F, G, H)(\alpha, \beta, \gamma)^2\}$$
$$\times \exp\{-2(\alpha U + \beta V + \gamma W)\}.\vartheta(U, V, W),$$

or say
$$= \exp\{-(A, \ldots)(\alpha, \beta, \gamma)^2\}$$
$$\times \exp\{-2(\alpha U + \beta V + \gamma W)\}.\vartheta(U, V, W).$$

Demonstration. Writing $\vartheta(U_1, V_1, W_1) = \Sigma . \exp \Theta_1$, then in the expression of Θ_1 we may in place of l, m, n write $l-\alpha, m-\beta, n-\gamma$; we thus obtain

$$\begin{aligned}\Theta_1 = (A, \ldots)(l-\alpha, m-\beta, n-\gamma)^2 \\ + \{(l-\alpha)[U + x\pi i + (A, \ldots)(\alpha, \beta, \gamma)] \\ + (m-\beta)[V + y\pi i + (H, \ldots)(\alpha, \beta, \gamma)] \\ + (n-\gamma)[W + z\pi i + (G, \ldots)(\alpha, \beta, \gamma)]\},\end{aligned}$$

which is

$$\begin{aligned}= (A, \ldots)(l, m, n)^2 \\ +2(lU+mV+nW)+2(lx+my+nz)\pi i+2(A,\ldots)(l,m,n)(\alpha,\beta,\gamma) \\ -2(A,\ldots)(l,m,n)(\alpha,\beta,\gamma) \\ -2(\alpha U+\beta V+\gamma W)-2(\alpha x+\beta y+\gamma z)\pi i-2(A,\ldots)(\alpha,\beta,\gamma)^2 \\ + (A,\ldots)(\alpha,\beta,\gamma)^2,\end{aligned}$$

which is

$$\begin{aligned}= (A, \ldots)(l, m, n)^2 + 2(lU + mV + nW) \\ - (A, \ldots)(\alpha, \beta, \gamma)^2 - 2(\alpha U + \beta V + \gamma W) \\ + 2[(l-\alpha)x + (m-\beta)y + (n-\gamma)z]\pi i.\end{aligned}$$

Hence, rejecting the last line, which (as an even multiple of πi) leaves the exponential unaltered, we see that $\vartheta(U_1, V_1, W_1)$ is $= \vartheta(U, V, W)$ into the factor

$$\exp\{-(A, \ldots)(\alpha, \beta, \gamma)^2\} . \exp\{-2(\alpha U + \beta V + \gamma W)\},$$

which is the theorem in question.

In many cases a formula which belongs to an indefinite number s of letters, is most easily intelligible when written out for three letters, but it is sometimes convenient to speak of the s letters $l, m, \ldots n$, or even the s letters $l, \ldots n$, and to write out the formulæ accordingly.

COMPUTATION OF THE CUBE ROOT OF 2.

By *Artemas Martin.*

In vol. v., p. 172, Mr. Gray has given the cube root of 2 to 28 places, which he states was computed by Horner's process, and in vol. vi., p. 106, he has extended the root to 32 places.

Having recently computed the value of the root to a greater number of places, by the method of approximation found in Simpson's *Algebra*, I submit it, with the work, for publication.

Let R = the true n^{th} root of a number N, and r = a near approximate root, and put $q = \dfrac{nr^n}{N - r^n}$; then (Simpson's *Algebra*, p. 169)

$$R = r + \frac{r(2q+n)}{q(2q+2n-1)+\frac{1}{6}(n-1)(2n-1)}, \text{ very nearly,}$$

which he says (p. 165) "quintuples the number of figures at every operation."

Taking $n = 3$ we have for the cube root of N,

$$R = r + \frac{r(2q+3)}{q(2q+5)+\frac{5}{3}}, \text{ very nearly,}$$

To compute the cube root of 2, take $r = 1{\cdot}25 = \frac{5}{4}$, then

$$\sqrt[3]{2} = 1{\cdot}25 + \frac{\frac{5}{4}(253)}{125(255)+\frac{5}{3}}, \quad = \frac{5}{4} + \frac{759}{76504},$$

$$= \frac{96389}{76504}, \quad = 1{\cdot}25992\ 10498+,$$

which is true to the last figure.

Now take $r = \frac{96389}{76504}$, then $r^3 = \frac{895534711311869}{447767355672064}$, and, after some reduction we get

$$\sqrt[3]{2} = \frac{96389}{76504} + \frac{5569174100732765858417747}{3681291779855685128169959391884736464}$$

$$= \frac{4638187004245850441098807007546772121}{3681291779855685128169959391884736464}$$

$$= 1{\cdot}25992\ 10498\ 94873\ 16476\ 72106\ 07278\ 22835\ 05702\ 51464\ 70150+,$$

which, if Simpson's remark is true and I have made no mistake, is correct to fifty places of decimals.

Erie, Pa., U.S.A.,
January 31, 1877.

VERIFICATION AND EXTENSION OF THE VALUE OF THE CUBE ROOT OF 2.

By *Peter Gray, F.R.A.S.*, &c.

I HAVE verified Mr. Martin's 50 decimal value of the cube root of 2, by direct multiplication by means of the arithmometer, and extended it to 56 decimals in the following manner:

The cube root was divided into six groups, or periods of eight figures, and in the work seven such periods, that is 56 decimals, were calculated. The work is given below, the method of formation being as explained in the *Messenger*, Vol. V. p. 172 (March, 1876).

1.25992104	98948731	64767210	60727822	83505702	51464701	50
1.58740102	70346816					
	1 24667588	06820024				
	1 24667588	06820024				
		81601570	58109840			
		81601570	58109840			
			76512260	65117488		
			76512260	65117488		
				1 05210590	90977008	
				1 05210590	90977008	
					64841459	60720904
		97908513	66510361		64841459	60720904
			64086332	39910510		62996052
			64086332	39910510		62996052
				60089409	23293882	
				60089409	23293882	
					82627832	44164162
					82627832	44164162
						50923669
				41947914	91184100	50923669
					39331716	00316620
					39331716	00316620
						54084313
						54084314
						36878684
1.58740105	19681994	74751705	63927230	82603914	93327899	83290125
1.99999998	18130920					
	24797758	34975376				
		94181245	90537320			
			80543262	10591920		
	1 57071319	48556755		1 04074409	23495056	
		19475083	29849614		1 17585783	56909496
			73965863	49836355		1 04938981
		1 02811537	15957050	63255182	84845130	
			12747478	38616740	81735524	65933134
				48414593	75593050	92346773
			96399408	40701310	41403883	30128300
				11952446	28237068	53500250
					45395082	35436510
				1 32557039	03578710	38821614
					16435587	25729788
						62421936
					81695120	40533605
						10129279
						79370053
1.99999999	99999999	99999999	99999999	99999999	99999999	96199719
					Error	3800281

It is only necessary to indicate very briefly the details of the process.

In the first operation, that is in the formation of the square, the first period of the root (in which the initial unit is included) is set on the machine, and multiplied by itself and the succeeding periods in order, all the products after the first being duplicated. The results are written down in black ink.*

The second period is now set on and dealt with in the same way; then the third, and finally the fourth; the colour of the ink being changed for each new setting of the machine. The initial product of each setting falls two periods to the right of that of the preceding setting. The sum of the results is of course $2^{\frac{2}{3}}$, which has to be multiplied by the root.

Again setting on the first period of $2^{\frac{1}{3}}$, it is multiplied in succession by all the periods of $2^{\frac{2}{3}}$. Setting on the second period it is multiplied as before, beginning with the first period of $2^{\frac{2}{3}}$, and necessarily stopping at the penultimate. And so with the rest, the colour of the ink being changed for each set of results.

My machine is a ten-figure one, and in consequence I am able to include the initial figure of the root in the first period. This is the cause of sundry gaps which appear in the work.

The result shows an error of 3800281 in the 51st and following places: denoting it by e and the correction to be applied to the root r by c, we have

$$c = \frac{e}{3r^2} = 798009.$$

giving for the last (the seventh) period of the corrected root 507 98009.

Availing ourselves of the previous work the verification of the extended root is very easy, especially as the correction extends only to one period.

The corrected seventh column is given below, and it will be seen that only three products in all had to be revised. These are marked by asterisks.

---

* The figures that are written in black ink are printed in modern type, those that are written in red ink in antique type.

Corrected seventh column.	Corrected seventh column. (*continued*).
60720904	
60720904	56909496
64001480*	107472501*
64001480*	
	65933134
	92346773
44164162	30128300
44164162	53500250
50923669	35436510
50923669	38821614
00316620	25729788
00316620	62421936
54084313	40533605
54084314	10129279
36878684	80636813*
85300981	99999999

Mr. Martin's fifty-place value is thus completely verified. It appears that the last figure, instead of 0, should *conventionally* be 1.

The fifty-six value found above is

$$\sqrt[3]{2} = 1.25992\ 10498\ 94873\ 16476\ 72106\ 07278\ 22835\ 05702\ 51464\ 70150\ 79800\ 9$$

The value of the cube root of 4 also occurs as the result of the first multiplication.

The value is

$$\sqrt[3]{4} = 1.58740\ 10519\ 68199\ 47475\ 17056\ 39272\ 30826\ 03914\ 93327\ 89985\ 30098\ 1.\dagger$$

London, *March 5*, 1877.

† [In connexion with this calculation of the cube root of 2 and of 4 to 56 decimal places I may mention two similar numerical calculations extending to a great number of places, that are but little known. The first is the value of the square root of 2, which was calculated by Mr. W. H. Colvill to 110 places, and was verified by actual multiplication by Mr. James Steel. The value is given in De Morgan's *Budget of Paradoxes* (1873) p. 293. The second is the value of the real root (2.09455...) of the celebrated cubic equation $x^3 - 2x - 5 = 0$, which, at the instigation of De Morgan, was calculated by Mr. J. P. Hicks to 152 places, as an example of the power of Horner's method. The result is given in vol. x. p. 337 (part ii.) of the *Transactions of the Cambridge Philosophical Society*. One hundred places had been previously calculated by Mr. W. H. Johnston (*Mathematician*, t. iii. p. 289).—J. W. L. G.].

TRANSACTIONS OF SOCIETIES.

London Mathematical Society.

Thursday, May 10th, 1877.—Lord Rayleigh, F.R.S., *President*, in the chair. Mr. Tucker communicated a short account of a paper by Dr. Hirst on the correlation of two planes. In a former paper on the subject (Proceedings, vol. v., p. 40) the nature and properties were described *first* of an ordinary correlation satisfying any eight given conditions; *secondly* of an exceptional correlation of the first order, possessing either a singular point or a singular line in each plane, and satisfying seven conditions; and *thirdly* of an exceptional correlation of the second order, having in each plane not only a singular point but also a singular line passing through that point, and satisfying six conditions. Moreover, the two following numerical relations were established between the (π, λ) exceptional correlations of the first order, with singular points and singular lines respectively, which satisfy any seven conditions, and the (μ, ν) ordinary correlations which, besides satisfying these same conditions, possess a given pair of conjugate points or conjugate lines respectively $(2\nu = \mu + \pi,\ 2\mu = \nu + \lambda)$. It was by means of these relations that the number of ordinary correlations was determined which satisfy any eight elementary conditions. Before they could be applied, however, the exceptional correlations of the first order which satisfy any seven elementary conditions had to be directly determined, and this determination not unfrequently necessitated the consideration of the projective properties of curves of high order. In the present paper the writer shows that the object just referred to can be attained in a very much simpler manner by means of two general relations, hitherto unobserved, connecting the number of exceptional correlations of the second order which satisfy any six conditions with the numbers of exceptional correlations of the first order, which, besides satisfying the six conditions in question, possess a given pair either of conjugate points or conjugate lines. The Secretary then read part of a paper by Prof. H. Lamb, of the University of Adelaide, "On the free motion of a solid through an infinite mass of liquid." Suppose that we have a solid body of any form immersed in an infinite mass of perfect liquid, that motion is produced in this system from rest by the action of any set of impulsive forces applied to the solid, and that the system is then left to itself. The equations of motion of a body under these circumstances have been investigated independently by Thomson and by Kirchhoff, and completely integrated for certain special forms of the body. The object of the present communication is, in the first place, to examine the various kinds of permanent or *steady* motion of which the body is capable, without making any restrictions as to its form or constitution; and, in the second, to show that when the initiating impulses reduce to a couple only, the complete determination of the motion can be made to depend upon equations identical in form with Euler's well-known equations of motion of a perfectly free rigid body about its centre of inertia, although the interpretation of the solution is naturally more complex. Free use is made throughout the paper of the ideas and the nomenclature of the theory of screws, as developed and established by Dr. Ball.

Herr Weichold (Head Master of the Johanneum, Zittau, Saxony) sent a paper (read in part by the Secretary) containing a solution of the irreducible case *i.e.*, of the problem to express the three roots of a complete equation of the third degree, in the case of *all* these roots being *real*, directly in terms of its coefficients by means of purely algebraical and really performable operations, whose number shall always be limited except in the case when all these roots are incommensurable. Mr. H. Hart made three communications: (1) "On the kinematic paradox." Prof. Sylvester has described a system of Peaucellier's cells, the poles of which all move in a straight line, but two of which, not directly connected, always remained at a constant distance. Such a result is very easily obtained by means of the following relations connecting six points, A, B, C, D, E, F, lying on a straight line (fig. 3). If:

$$\left.\begin{aligned} AB.AC &= a^2 \\ BC.BD &= 4a^2 \\ EB.ED &= a^2 \\ FA.FE &= 2a^2 \end{aligned}\right\}, \text{ then } FB = a.$$

He then spoke on the solution of the algebraical equation $f(x) = 0$ by linkwork, considering three points, the preparation of the equation (put under the form $\frac{A}{x+a} + \frac{B}{x+b} + \dots = k$), the representation of the terms of this equation, and the method of adding these terms. He shewed that for the solution of the cubic $x^3 + px^2 + qx + r = 0$, treated under the form

$$x + p + \frac{\left(q - \frac{r}{p}\right) x}{x^2 + \frac{r}{p}} = 0,$$

two reciprocators alone are required. He then spoke on the production of circular and rectilinear motion. The particular problem considered, he thus enunciated "to find if possible the relation that must exist between the fourteen segments of the bars placed as in (fig. 4) in order that the system may be capable of free motion. He showed that seven equations can be obtained connecting the fourteen quantities only, so that any seven being given, the remaining seven can be determined in terms of them. Mr. Hart then proceeded to an application to the cases of five-bar motion laid before the Society at its April meeting. Mr. Kempe stated that the cases submitted by Mr. Hart at the previous meeting had also occupied some of his attention, and he proceeded to remark that he had determined the positions that the lines GE, KM must have, and that the determination of one involved the determination of the other, as the position of either turned upon the fact that the angles at A and H must be equal. Prof. Cayley also made a few remarks on the subject.

Mr. J. W. L. Glaisher stated that he had had all the cases in which there are more than 50 consecutive composite numbers looked out from Burckhardt's and Dase's tables, which extend from 1 to 3,000,000, and from 6,000,000 to 9,000,000. A list of the groups of composite numbers so found in the first three millions was exhibited to the Society. The two most noteworthy stretches were a group of 111 consecutive composite numbers between 370,261 and 370,373, and of 113 between 492,113 and 492,227. These two very long sets of numbers without a prime occurring in the first half million were remarkable, as he thought they exceeded what one would *à priori* expect. The longest stretch noted in the three millions occurred in the third million and consisted of 147 consecutive composite numbers, and the next largest occurred in the second million, where there were two stretches, each of 131. The 147 consecutive composite numbers were those between 2,010,733 and 2,010,881, and the stretches of 131 were from 1,357,201 to 1,357,333, and from 1,561,919 to 1,562,051. Questions were put to the meeting for information by Profs. Cayley and Clifford.

Thursday, June, 14th, 1877.—Lord Rayleigh, F.R.S., *President*, in the chair. Prof. Crofton, F.R.S., proved some geometrical theorems relating to mean values. These theorems were chiefly interesting as examples of the employment of the theory of probability to establish mathematical results; they were of a kindred nature with theorems given in the *Phil. Trans.*, 1868, p. 185, and in Williamson's "Integral Calculus," second edition, p. 329. Mr. Merrifield made a few remarks on the communication. Prof. Clifford, F.R.S., read a paper on the canonical form and dissection of a Riemann's surface. The object of the paper is to assist students of the theory of complex functions by proving the chief propositions about Riemann's surfaces in a concise and elementary manner. To this end certain results of Puiseux's were assumed at the outset. Prof. Smith, in making remarks on the paper, expressed his indebtedness to the author in having cleared up a difficulty which presents itself in Lüroth's paper on the subject. Prof. H. J. S. Smith, F.R.S., gave a short account of a further communication upon Eisenstein's theorem. Mr. Tucker communicated a paper by Mr. J. C. Malet entitled, "Proof that every algebraic equation has a root." The Society's next meeting will be held on the second Thursday in November.

R. Tucker, M.A., *Hon. Sec.*

MATHEMATICAL NOTES.

On a pair of Algebraical Equations.

Consider the equations

$$\begin{aligned} x^2 + y^2 + z^2 &= 1, \\ a_1x + b_1y + c_1z - xu &= 0, \\ a_2x + b_2y + c_2z - yu &= 0, \\ a_3x + b_3y + c_3z - zu &= 0, \end{aligned}$$

whence we have

$$u = \frac{\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}}{\begin{vmatrix} x, & y, & z, & 0 \\ a_1, & b_1, & c_1, & x \\ a_2, & b_2, & c_2, & y \\ a_3, & b_3, & c_3, & z \end{vmatrix}} = \frac{\nabla}{U} \ldots\ldots\ldots\ldots \ (1),$$

$$x = \frac{\begin{vmatrix} 1, & y, & z, & 0 \\ 0, & b_1, & c_1, & x \\ 0, & b_2, & c_2, & y \\ 0, & b_3, & c_3, & z \end{vmatrix}}{U}$$

$$= \frac{xA_1 + yA_2 + zA_3}{U},$$

A_1, A_2, A_3 being the minors of a_1, a_2, a_3 in ∇. Thus

$$x(A_1 - U) + y\,A_2 + z\,A_3 = 0,$$

and similarly

$$\begin{aligned} x\,B_1 + y(B_2 - U) + z\,B_3 &= 0, \\ x\,C_1 + y\,C_2 + z(C_3 - U) &= 0, \end{aligned}$$

$B_1, C_1 \ldots$ being the minors of $b_1, c_1 \ldots$ in ∇, whence we see that U is a root of the equation

$$\begin{vmatrix} A_1 - U, & A_2\ , & A_3 \\ B_1\ , & B_2 - U, & B_3 \\ C_1\ , & C_2\ , & C_3 - U \end{vmatrix} = 0, \ldots\ldots\ldots\ldots\ldots\ldots \ (2).$$

Also u is a root of the equation

$$\begin{vmatrix} a_1-u, & a_2\,, & a_3 \\ b_1\,, & b_2-u, & b_3 \\ c_1\,, & c_2\,, & c_3-u \end{vmatrix} = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

The equations (2) and (3) are therefore such, that their roots are connected by the curious relation (1), viz.

$$uU = \nabla.$$

The direct verification is easy, for the equation (3) is

$$u^3 - (a_1+b_2+c_3)\,u^2 + (A_1+B_2+C_3)\,u - \nabla = 0,$$

and, in virtue of the well-known values of the reciprocal determinant and its minors, the equation (2) is

$$U^3 - (A_1+B_2+C_3)\,U^2 + (a_1\nabla + b_2\nabla + c_3\nabla)\,U - \nabla^2 = 0,$$

that is, on multiplying ∇ and dividing by U^3,

$$\left(\frac{\nabla}{U}\right)^3 - (a_1+b_2+c_3)\left(\frac{\nabla}{U}\right)^2 + (A_1+B_2+C_3)\,\frac{\nabla}{U} - 1 = 0.$$

PAUL MANSION.

On a Discontinuous Series.

We have identically

$$x = (x - x^2 - x^3) + (x^2 - x^4 - x^5) \ldots + (x^{n-1} - x^{2n-2} - x^{2n-1}) + x^n\,\frac{1-x^n}{1-x},$$

whence, for $x < 1$,

$$x = (x - x^2 - x^3) + (x^2 - x^4 - x^5) + (x^3 - x^6 - x^7) + \&c.$$

Integrate the first equation between the limits 0 and x, x being < 1, and

$$\frac{x^2}{2} = \left(\frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4}\right) + \left(\frac{x^3}{3} - \frac{x^5}{5} - \frac{x^6}{6}\right) \ldots$$
$$+ \left(\frac{x^n}{n} - \frac{x^{2n-1}}{2n-1} - \frac{x^{2n}}{n}\right) + \int_0^x x^n\,\frac{1-x^n}{1-x}\,dx,$$

When $n = \infty$, the integral just written vanishes, and therefore

$$\tfrac{1}{2}x^2 = (\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4) + (\tfrac{1}{3}x^3 - \tfrac{1}{5}x^5 - \tfrac{1}{6}x^6) + (\tfrac{1}{4}x^4 - \tfrac{1}{7}x^7 - \tfrac{1}{8}x^8) + \&c.$$

Although both sides of this equation remain finite when $x = 1$, yet we may not give x this value; for when $x = 1$, the series changes its value abruptly from a value as near as we please to $\frac{1}{2}$ to $\frac{1}{2} - \log 2$. This is evident, for we know that

$$\tfrac{1}{2} - [1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \tfrac{1}{7} - \tfrac{1}{8} + \&c.] = \tfrac{1}{2} - \log 2.$$

We may conclude from this, that

$$\log 2 = \lim_{n=\infty} \int_0^1 x^n \frac{1-x^n}{1-x}\, dx,$$

which it is easy to verify directly; and we also see that the statement contained in most French treatises on the Integral Calculus, that the integral of a convergent series, if itself convergent, is a continuous function of the independent variable with respect to which the integration is effected, even when the series integrated ceases to be convergent for the superior limit of integration, is not true. PAUL MANSION.

On the determination of the Sign of any term of a Determinant.

The following seems to me to be preferable to the usual statement of the rule for determining the sign of any term of a determinant. Denoting the disarranged subscripts by $\alpha, \beta \ldots$, suppose that the original place of the subscript α is occupied by β, that of β by γ, and so on until we come to λ, whose place is occupied by α. Call $\alpha, \beta, \gamma \ldots \lambda$ a cycle, and let all the subscripts be thus grouped into cycles; let there be m cycles, some of which of course may consist of single subscripts. Then an interchange of two subscripts will increase or decrease the number m by unity. For, first, let the interchanged subscripts be ε and ρ, belonging to the two cycles

$$\alpha \ldots \delta\varepsilon \ldots \lambda \text{ and } \mu \ldots \pi\rho \ldots \omega,$$

then the interchange converts these two cycles into the single cycle

$$\alpha \ldots \delta\rho \ldots \omega\mu \ldots \pi\varepsilon \ldots \lambda.$$

Secondly, if the interchanged subscripts belong to the same cycle, as ε and ρ in

$$\alpha \ldots \delta\varepsilon \ldots \pi\rho \ldots \omega,$$

the interchange converts this into the two cycles

$$\alpha \ldots \delta\rho \ldots \omega \text{ and } \varepsilon \ldots \pi.$$

Thus, n denoting the order of the determinant, if we give to each term the positive or negative sign according as $n-m$ is even or odd, an odd number of interchanges will always change the sign of the term, and it is easy to see that $n-m$ is the least number of interchanges that will convert the given term into the principal term or diagonal.

W. WOOLSEY JOHNSON.

Annapolis Md., U.S.A.,
Dec. 1, 1876.

Summation of a Series.

If $1^p + 2^p + 3^p + \dots n^p$ be denoted by $S_{n,p}$, to find $S_{n,p+1}$ in terms of $S_{n,p}$, $S_{n,p-1}$, &c., and so calculate generally the sums of the powers of the natural numbers.

Let AB, BC, CD, &c. represent 1^p, 2^p, 3^p, &c. respectively, draw BF, CG, DH, &c. at right angles to AE and equal to 1, 2, 3, &c. respectively. Then the rectangles AF, BG, CH, &c. will represent 1^{p+1}, 2^{p+1}, 3^{p+1}, &c.

Suppose DK is the n^{th}, produce EK to L, $(KL = 1)$ and complete the rectangles as in fig. 5.

Then $AE = S_{n,p}$, $EL = n + 1$, and we have $S_{n,p+1}$ + series of rectangles gF, hG, kH, &c. $= (n+1)\, S_{n,p}$.

In the series of rectangles gF, hG, &c., the $r^{\text{th}} = S_{r,p}$.

Hence $$S_{n,p+1} + \Sigma . S_{r,p} = (n+1)\, S_{n,p}.$$

In the term $\Sigma . S_{r,p}$, $S_{r,p}$ must be expressed in terms of r, and then all values from 1 to n given to r. The result will give the relation between $S_{n,p+1}$ and the sums of the lower powers $S_{n,p}$, $S_{n,p-1}$, &c.

Ex. Let $p = 1$,

$$S_{n,2} + \Sigma\left(\tfrac{1}{2}r^2 + \tfrac{1}{2}r\right) = (n+1)\, S_{n,1},$$

$$\tfrac{3}{2}S_{n,2} = \left(n + \tfrac{1}{2}\right) S_{n,1} \text{ or } S_{n,2} = \tfrac{1}{6}n(n+1)(2n+1)$$
$$= \tfrac{1}{3}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{6}n.$$

$p = 2$,

$$S_{n,3} + \Sigma\left(\tfrac{1}{3}r^3 + \tfrac{1}{2}r^2 + \tfrac{1}{2}r\right) = (n+1)\, S_{n,2},$$

$$\tfrac{4}{3}S_{n,3} = \left(n + \tfrac{1}{2}\right)\left(\tfrac{1}{3}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{6}n\right) - \tfrac{1}{2}\left(\tfrac{1}{2}n^2 + \tfrac{1}{2}n\right),$$

which gives $S_{n,3} = \tfrac{1}{4}n^4 + \tfrac{1}{2}n^3 + \tfrac{1}{4}n^2 = \left\{\tfrac{1}{2}n(n+1)\right\}^2$.

$p = 3$,

$$S_{n,4} + \Sigma\left(\tfrac{1}{4}r^4 + \tfrac{1}{2}r^3 + \tfrac{1}{4}r^2\right) = (n+1)\, S_{n,3},$$

$$\tfrac{5}{4}S_{n,4} = \left(n + \tfrac{1}{2}\right) S_{n,3} - \tfrac{1}{4}S_{n,2}$$
$$= \left(n + \tfrac{1}{2}\right)\left(\tfrac{1}{4}n^4 + \tfrac{1}{2}n^3 + \tfrac{1}{4}n^2\right) - \tfrac{1}{4}\left(\tfrac{1}{3}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{6}n\right),$$

which gives $S_{n,4} = \tfrac{1}{5}n^5 + \tfrac{1}{2}n^4 + \tfrac{1}{3}n^3 - \tfrac{1}{30}n$.

$p = 4$,

$$S_{n,5} + \Sigma\left(\tfrac{1}{5}r^5 + \tfrac{1}{2}r^4 + \tfrac{1}{3}r^3 - \tfrac{1}{30}r\right) = (n+1)\, S_{n,4},$$

$$\tfrac{6}{5}S_{n,5} = \left(n + \tfrac{1}{2}\right) S_{n,4} - \tfrac{1}{3}S_{n,3} + \tfrac{1}{30}S_{n,1},$$

which gives $S_{n,5} = \tfrac{1}{6}n^6 + \tfrac{1}{2}n^5 + \tfrac{5}{12}n^4 - \tfrac{1}{12}n^2$,

and so on for higher values of p.

To show that the two first terms in $S_{n,p}$ are $\dfrac{n^{p+1}}{p+1}+\frac{1}{2}n^p$. Assume this to be true for (p), then

$$S_{n,p+1}+\Sigma\left(\frac{r^{p+1}}{p+1}+\tfrac{1}{2}r^p+\&c.\right)=(n+1)\,S_{n,p},$$

$$S_{n,p+1}\left(1+\frac{1}{p+1}\right)=(n+\tfrac{1}{2})\left(\frac{n^{p+1}}{p+1}+\tfrac{1}{2}n^p+\&c.\right),$$

$$S_{n,p+1}=\frac{n^{p+2}}{p+2}+\tfrac{1}{2}n^{p+1}+\&c.$$

If therefore the law be true for (p) it is so for $(p+1)$.

J. M. CROKER.

Solution of a Cubic Equation.

Substituting for x, $y+n$, we get

$$y^3+3(n+p)\,y^2+3(n^2+2pn+q)\,y+n^3+3pn^2+3qn+r=0.$$

This may be put in the form

$$my^3=(ay+b)^3,$$

if $$(n+p)(n^3+3pn^2+3qn+r)=(n^2+2pn+q)^2,$$

which reduces to

$$(p^2-q)\,n^2+(pq-r)\,n+(q^2-pr)=0.$$

It will be found (most easily by making $p=0$ in the usual way), that if the roots of the original equation are all real, the roots of the quadratic for n are imaginary, and we have to extract the cube root of a binomial of the form $\alpha+\beta i$ to determine b and m. Thus the method presents no particular advantage over Cardan's. It is perhaps interesting on account of the coefficients in the auxiliary equation for n.

R. W. GENESE.

On the Theory of Envelopes.

The envelope of $\phi(\alpha\beta\gamma\ldots\lambda)=0$, $\alpha\beta\gamma\ldots$ being coordinates and λ a variable, is generally to be obtained by eliminating λ between this equation and $\dfrac{d\phi}{d\lambda}=0$.

Let $R=0$ be the result of the elimination. The logical conclusion from the theory is that $\phi(\lambda)=0$ meets $\phi(\lambda+d\lambda)=0$ where it is met by $R=0$. If then $\phi=0$ passes through

fixed points, $R=0$ will in general contain factors $P=0$, $Q=0$, &c. passing through the fixed points. These factors therefore must not be regarded as forming part of the envelope.

Ex. To find the envelope of the parabola circumscribing a triangle ABC.

Taking CA, CB as axes, the equation to any one of the parabolas is

$$x(x-a)+\lambda^2 y(y-b)+2\lambda xy=0,$$

And the envelope seems to be

$$x^2y^2=x(x-a)(y)(y-b),$$

i.e. $$x=0,\ y=0,\ \text{and}\ \frac{x}{a}+\frac{y}{b}=1,$$

or AB, AC, BC, which is obviously absurd.

The envelope is, we know, the line at infinity which may perhaps be inferred from the fact that the envelope appears in the form of the fourth degree, and reduces to three linear factors only, the other being the line at infinity. This line appears clearly if the problem be treated by the method of Trilinear Coordinates.

R. W. GENESE.

[The example, although simple, is an instructive one Introducing z, μ for homogeneity, the equation is

$$\lambda^2 y(y-bz)+2\lambda\mu xy+\mu^2 x(x-az)=0,$$

giving the envelope

$$xy[(x-az)(y-bz)-xy]=0;$$

that is, $$xy(bx+ay-abz)z=0;$$

viz. we have thus the four lines

$$x=0,\quad y=0,\quad \frac{x}{a}+\frac{y}{b}-z=0,\quad z\doteq 0.$$

Writing these values successively in the equation of the curve, we find respectively

$$\lambda^2 y(y-bz)=0,$$
$$\mu^2 x(x-az)=0,$$
$$(b\lambda-a\mu)^2\frac{xy}{ab}=0,$$
$$(\lambda y+\mu x)^2=0;$$

viz. in each case the equation in λ, μ has (as it should have) two equal roots; but in the first three cases the values are *constant;* viz. we find $\lambda = 0$, $\mu = 0$, $b\lambda - a\mu = 0$ respectively; and the curves $x = 0$, $y = 0$, $\frac{x}{a} + \frac{y}{b} - z = 0$ are for this reason not proper envelopes.

It is to be remarked that writing in the equation of the parabola these values $\lambda = 0$, $\mu = 0$, $b\lambda - a\mu = 0$ successively, we find respectively

$$x(x - az) = 0,$$

$$y(y - bz) = 0,$$

$$(bx + ay)(bx + ay - abz) = 0;$$

viz. in each case the parabola reduces itself to a pair of lines, one of the given lines and a line parallel thereto through the intersection of the other two lines; the parabola thus becomes a curve having a dp on the line at infinity.

In the fourth case $z = 0$, the equation in λ, μ is $(\lambda y + \mu x)^2 = 0$, giving a variable value $\lambda \div \mu = -x \div y$; hence $z = 0$, the line at infinity is a proper envelope.

The true geometrical result is that the envelope consists of the three points A, B, C, and the line infinity; a point *quâ* curve of the order 0 and class 1 is not representable by a single equation in point-coordinates, and hence the peculiarity in the form of the analytical result. A. CAYLEY.]

An Arithmetical Theorem.

In vol. XXVI. of the Mathematical Reprint of the *Educational Times* Mr. Martin proposed the following problem (Question 5009, page 28. "Prove (1) that all powers of 12890625 terminate with 12890625, and (2) that all powers of numbers terminating with 12890625 terminate with 12890625." In the following note an attempt is made to answer the question: How many numbers, of a specified number of digits, exist in any given scale of notation, which have the properties here predicated of 12890625?

The work is much simplified by observing that the properties in question belong to every number whose square terminates in the number itself. Hence, if N be a number, in the scale of r, consisting of n digits, we have to find the number of solutions of

$$N^2 - N = Kr^n \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

If A, B be two co-factors of r^n (1) may be replaced by

$$\left.\begin{aligned} N &= hA \\ N-1 &= kB \end{aligned}\right\} \dots\dots\dots\dots\dots\dots (2).$$

Here A, B must be prime to each other, and N being less than r^n or AB, we have

$$h < B, \; k < A,$$

and from (2) $$hA - kB = 1.$$

If we regard A, B as known, and h', k' as particular solutions of this last equation, the general solution (in integers) is

$$h = h' + Bt, \; k = k' + At,$$

t being any integer. It is clear that for one and only one value of t, we have

$$0 < h < B, \; 0 < k < A,$$

so that A, B being determined there is one and only one solution of (2) or (1).

It only remains to determine how many values A may have. Let

$$r^n = a_1 . a_2 ... a_q,$$

where a_1, $a_2 ... a_q$ are powers of primes. Suppose $a_1 = \alpha^m$ where α is a prime. Then A, B being prime to each other, A must be a multiple of α^m (or a_1), or it cannot contain α at all. There are thus 2^q forms of A, since either of the q quantities a may be included in A or not. Of these 2^q values two are inadmissible, viz. when all the a's are included in A (or $A = r^n$), and when they are all excluded (or $B = r^n$), for these are inconsistent with the fact that N is less than r^n. Hence there are $2^q - 2$ values of N, and no more, which satisfy the conditions of Mr. Martin's theorem.

These numbers are given to ten digits each in the cases $r = 6, 10, 12$ in my solution of question 5277 of the *Educational Times*. They are

$r = 6$, 3221350213, 3334205344,

$r = 10$, 8212890625, 1787109376,

$r = 12$, 21e61e3854 , 9t05t08369 , ($t = 9 + 1$, $e = 9 + 2$).

The above note will apply to numbers such that the square of each ends in its arithmetical complement, for here we start with the equation

$$N^2 + N = Kr^n.$$

H. W. LLOYD TANNER.

On a Class of Definite Integrals.

If
$$u=\int_\beta^\alpha R\cos bx\,dx \text{ .... (1)},$$
where R is a function of x and t, then, by integration by parts,
$$b^2u=\left[bR\sin bx+\frac{dR}{dx}\cos bx\right]_\beta^\alpha-\int_\beta^\alpha\frac{d^2R}{dx^2}\cos bx\,dx.$$
Differentiating (1) with respect to t,
$$\frac{d^2u}{dt^2}=\int_\beta^\alpha\frac{d^2R}{dt^2}\cos bx\,dx.$$
If, then, R be such that
$$\frac{d^2R}{dx^2}+\frac{d^2R}{dt^2}=0 \text{ .... (2)},$$
then
$$\frac{d^2u}{dt^2}-b^2u+\left[bR\sin bx+\frac{dR}{dx}\cos bx\right]_\beta^\alpha=0 \text{...... (3)}.$$
Now from (2) we have
$$R=F(x+it)+f(x-it),$$
and this value of R being substituted in (3), we have
$$\frac{d^2u}{dt^2}-b^2u+\phi(t)=0,$$
of which the solution is
$$2bu=Ae^{bt}+Be^{-bt}-e^{bt}\int e^{-bt}\phi(t)\,dt+e^{-bt}\int e^{bt}\phi(t)\,dt,$$
so that, except that the arbitrary constants have to be determined, we obtain the value of the definite integral
$$\int_\beta^\alpha\{F(x+it)+f(x-it)\}\cos bx\,dx,$$
expressed in terms of indefinite integrals with regard to t.

As an example, let
$$f(x+it)=\frac{i}{x+it},\quad F(x-it)=-\frac{i}{x-it},$$
then
$$f(x+it)+F(x-it)=\frac{2t}{x^2+t^2},$$
and
$$\frac{d}{dx}\{f(x+it)+F(x-it)\}=-\frac{4tx}{(x^2+t^2)^2}.$$

Thus the quantity in brackets in (3) vanishes both when $x=0$ and when $x=\infty$, so that, putting $\alpha=\infty$ and $\beta=0$, we have

$$\frac{d^2u}{dt^2}-b^2u=0,$$

and therefore the value of the definite integral

$$2t\int_0^\infty \frac{\cos bx}{x^2+t^2}\,dx$$

satisfies the differential equation, and is therefore of the form $Ae^{bt}+Be^{-bt}$. If the limits be α and β, so that the integral is

$$\frac{d^2u}{dt^2}-b^2u=4t\left\{\frac{\alpha\cos b\alpha}{(\alpha^2+t^2)^2}-\frac{\beta\cos b\beta}{(\beta^2+t^2)^2}\right\}-2bt\left\{\frac{\sin b\alpha}{\alpha^2+t^2}-\frac{\sin b\beta}{\beta^2+t^2}\right\}.$$

Similarly, if

$$f(x+it)=\frac{1}{x+it},\quad F(x-it)=\frac{1}{x-it},$$

we find that the value of the integral

$$2\int_\beta^\alpha \frac{x\cos bx}{x^2+t^2}\,dx$$

satisfies the differential equation

$$\frac{d^2u}{dt^2}-b^2u$$

$$=-2\left\{\frac{(t^2-\alpha^2)\cos b\alpha}{(t^2+\alpha^2)^2}-\frac{(t^2-\beta^2)\cos b\beta}{(t^2+\beta^2)^2}\right\}-2b\left\{\frac{\alpha\sin b\alpha}{t^2+\alpha^2}-\frac{\beta\sin b\beta}{t^2+\beta^2}\right\}.$$

The above process is a generalization of the method employed by Mr. Glaisher for the evaluation of the integral

$$\int_0^\infty \frac{\cos bx}{a^2+x^2}\,dx,$$

(*Messenger*, vol. I., p. 35, 1871), and was suggested by it.

ROBERT RAWSON.

Havant, 1876.

Theorem relating to the difference between the sums of the even and uneven divisors of a Number.

If $f(n)$ denote the sum of the uneven divisors of any number n, and $F(n)$ the sum of the even divisors of n (unity and n being both included as divisors), then

$$f(n)+f(n-1)+f(n-3)+f(n-6)+f(n-10)+\&c.$$
$$=F(n)+F(n-1)+F(n-3)+F(n-6)+F(n-10)+\&c.,$$

where 1, 3, 6, 10 ... are the triangular numbers, and it is supposed that

$$f(n-n)=0,\ F(n-n)=n.$$

Examples.

I. $n=14$,

$$\begin{aligned}&8+14+12+\ \ 1+1\\=\ &16+\ .\ +\ .\ +14+6.\end{aligned}$$

II. $n=15$,

$$\begin{aligned}&24+\ 8+\ 4+13+6+\ 0\\=\ &.\ +16+24+\ .\ +\ .+15.\end{aligned}$$

The theorem may also evidently be enunciated as follows:

If $\theta(n)$ denote the excess of the sum of the uneven divisors over the sum of the even divisors of n, then

$$\theta(n)+\theta(n-1)+\theta(n-3)+\theta(n-6)+\theta(n-10)+\&c.=0,$$

where it is supposed that

$$\theta(n-n)=-n.$$

J. W. L. GLAISHER.

On some Continued Fractions.

I. If in the two expansions

$$\frac{\log\{x+\sqrt{(1+x^2)}\}}{\sqrt{(1+x^2)}}=x-\tfrac{2}{3}x^3+\tfrac{2}{3}.\tfrac{4}{5}x^5-\tfrac{2}{3}.\tfrac{4}{5}.\tfrac{6}{7}x^7+\&c.,$$

$$\frac{\text{arc sin}\,x}{\sqrt{(1-x^2)}}=x+\tfrac{2}{3}x^3+\tfrac{2}{3}.\tfrac{4}{5}x^5+\tfrac{2}{3}.\tfrac{4}{5}.\tfrac{6}{7}x^7+\&c.,$$

the series be converted into continued fractions, we find

$$\begin{aligned}\frac{\log\{x+\sqrt{(1+x^2)}\}}{\sqrt{(1+x^2)}}&=\frac{x.}{1+}\ \frac{1.2x^2}{3-2x^2+}\ \frac{3.4x^2}{5-4x^2+}\ \frac{5.6x^2}{7-6x^2+\&c.}\\&=\frac{1}{x^{-1}+}\ \frac{1.2}{3x^{-1}-2x+}\ \frac{3.4}{5x^{-1}-4x+}\ \frac{5.6}{7x^{-1}-6x+\&c.},\end{aligned}$$

$$\begin{aligned}\frac{\text{arc sin}\,x}{\sqrt{(1-x^2)}}&=\frac{x}{1-}\ \frac{1.2x^2}{2x^2+3-}\ \frac{3.4x^2}{4x^2+5-}\ \frac{5.6x^2}{6x^2+7-\&c.}\\&=\frac{1}{x^{-1}-}\ \frac{1.2}{2x+3x^{-1}-}\ \frac{3.4}{4x+5x^{-1}-}\ \frac{5.6}{6x+7x^{-1}-\&c.};\end{aligned}$$

and, putting $x = \frac{1}{\sqrt{2}}$, we have

$$\frac{2}{\sqrt{3}} \log \frac{1+\sqrt{3}}{\sqrt{2}} = \frac{1}{1+} \; \frac{1}{2+} \; \frac{6}{3+} \; \frac{15}{4+} \; \frac{28}{5+\&c.},$$

$$\frac{\sqrt{2}}{\sqrt{3}} \log \frac{1+\sqrt{3}}{\sqrt{2}} = \frac{1}{\sqrt{2}+} \; \frac{1.2}{2\sqrt{2}+} \; \frac{3.4}{3\sqrt{2}+} \; \frac{5.6}{4\sqrt{2}+} \; \frac{7.8}{5\sqrt{2}+\&c.},$$

$$\tfrac{1}{2}\pi = \frac{1}{1-} \; \frac{1}{4-} \; \frac{6}{7-} \; \frac{15}{10-} \; \frac{28}{13-\&c.},$$

$$\frac{\pi}{2\sqrt{2}} = \frac{1}{\sqrt{2}-} \; \frac{1.2}{4\sqrt{2}-} \; \frac{3.4}{7\sqrt{2}-} \; \frac{5.6}{10\sqrt{2}-} \; \frac{7.8}{13\sqrt{2}-\&c.},$$

where 1, 6, 15, 28, ... are the alternate triangular numbers.

Putting $x = \frac{1}{2}$, we have

$$\frac{2}{\sqrt{5}} \log \frac{1+\sqrt{5}}{2} = \frac{1}{2+} \; \frac{1.2}{5+} \; \frac{3.4}{8+} \; \frac{5.6}{11+} \; \frac{7.8}{14+\&c.},$$

$$\frac{\pi}{3\sqrt{3}} = \frac{1}{2-} \; \frac{1.2}{7-} \; \frac{3.4}{12-} \; \frac{5.6}{17-} \; \frac{7.8}{22-\&c.}.*$$

If we take $x = 1$ in the first continued fraction, there results the equation

$$\frac{\log(1+\sqrt{2})}{\sqrt{2}} = \frac{1}{1+} \; \frac{1.2}{1+} \; \frac{3.4}{1+} \; \frac{5.6}{1+} \; \frac{7.8}{1+\&c.},$$

which may be compared with the continued fraction in the formula for $\frac{1}{2}\pi$,

$$\tfrac{1}{2}\pi - 1 = \frac{1}{1+} \; \frac{1.2}{1+} \; \frac{2.3}{1+} \; \frac{3.4}{1+} \; \frac{4.5}{1+\&c.}.$$

II. $$\frac{1 + x + x^3 + x^6 + x^{10} + x^{15} + x^{21} + \&c.}{1 - x - x^3 + x^6 + x^{10} - x^{15} - x^{21} + \&c.}$$

$$= \frac{1}{1-} \; \frac{2x}{1+x-} \; \frac{x^2-x^4}{1+x^3-} \; \frac{x^3-x^7}{1+x^5-} \; \frac{x^4-x^{10}}{1+x^7-\&c.},$$

the $n+1^{\text{th}}$ partial fraction being

$$\frac{x^n - x^{3n-2}}{1+x^{2n-1}};$$

this continued fraction is derived from the series in VI., p. 111., vol. V. (November, 1875).

J. W. L. Glaisher.

* See p. 79.

A simple proof of a Theorem relating to the Potential.

The value of a function, which satisfies Laplace's equation within a closed surface, is determined by the values at the surface. If two surface distributions be superposed, the value at any internal point is the sum of those due to the two surface distributions considered separately. By means of this principle a very simple proof may be given of the known theorem that the value of the potential at the centre of a sphere is the mean of those distributed over the surface.

On account of the symmetry it is clear that the central value would not be affected by any rotation of the sphere, to which the surface values are supposed to be rigidly attached.

Thus, if we conceive the sphere to be turned into n different positions taken at random and the resulting surface distributions to be superposed, we obtain a new surface distribution, whose mean value is n times greater than before, which determines a central value which is also n times greater than that due to the original distribution. When n is made infinite, the surface distribution becomes constant, in which case the central value is the same as the surface value, from which it follows that in the original state of things the central value was the mean of the surface values.

RAYLEIGH.

An Identity.

The following remarkable identity is given under a slightly different form by Gauss, *Werke*, t. iii., p. 424,

$$1+\left(\frac{\frac{1}{2}}{1}\right)^3 x+\left(\frac{\frac{1}{2}\cdot\frac{3}{2}}{1\,.\,2}\right)^3 x^2+\left(\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{1\,.\,2\,.\,3}\right)^3 x^3+\&\text{c.}$$

$$=\left\{1+\left(\frac{\frac{1}{4}}{1}\right)^2 x+\left(\frac{\frac{1}{4}\cdot\frac{5}{4}}{1\,.\,2}\right)^2 x^2+\left(\frac{\frac{1}{4}\cdot\frac{5}{4}\cdot\frac{9}{4}}{1\,.\,2\,.\,3}\right)^2 x^3+\&\text{c.}\right\}^2.$$

A. CAYLEY.

ON COGNATE RICCATIAN EQUATIONS.

By *Robert Rawson.*

1. IT is readily seen that the equation

$$\frac{du}{dx}+\frac{au}{x}+bx^n u^2-cx^m=0\ldots\ldots\ldots\ldots\ldots(1)$$

is transformed into

$$\frac{dy}{dx}-\frac{a+\alpha}{x}y+bx^{m-\alpha}y^2-cx^{n+\alpha}=0\ldots\ldots\ldots(2),$$

if $$buy=cx^\alpha\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

After considerable hesitation the system (1), (2), and (3) has been designated a cognate Riccatian, a term suggested by Sir J. Cockle's papers "On Linear Differential Equations of the Second Order" (*Messenger*, vols. I. and II.).

2. We also have

$$\frac{dy}{dx}+\frac{a-\alpha-2n-2}{x}y+bx^{m-\alpha}y^2-cx^{n+\alpha}=0\ldots\ldots(4),$$

if
$$u=\frac{cx^{\alpha}}{by}+\frac{n+1-a}{b}x^{-n-1}\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

and both of these transformations are particular cases of the following:

$$\frac{dy}{dx}-\left\{\frac{a+\alpha+2bAx^{\beta+n+1}}{x}\right\}y$$
$$+\left\{bx^{m-\alpha}-\frac{bA(a+\beta)}{c}x^{\beta-x-1}-\frac{b^2A^2}{c}x^{2\beta+n-\alpha}\right\}y^2-cx^{n+\alpha}=0\ldots(6),$$

where
$$u=\frac{cx^{\alpha}}{by}+Ax^{\beta}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

3. In (2) take $\alpha=0$, then it becomes

$$\frac{dy}{dx}-\frac{ay}{x}+bx^{m}y^2-cx^{n}=0\ldots\ldots\ldots\ldots\ldots(8),$$

where
$$buy=c\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

Again, in (4) take $\alpha=2a-2n-2$, and it becomes

$$\frac{dy}{dx}-\frac{ay}{x}-bx^{m+2n+2-2a}y^2-cx^{2a-n-2}=0\ldots\ldots\ldots(10),$$

where
$$u=\frac{cx^{2a-2n-2}}{by}+\frac{n+1-a}{b}x^{-n-1}\ldots\ldots\ldots\ldots(11).$$

It is easily seen from (8) and (10) that if $\phi(a, n, m)$ be the condition of solubility of (1), then $\phi(-a, m, n)$ and $\phi(-a,\ m+2n+2-2a,\ 2a-n-2)$ are also conditions of solubility of the same differential equation.

4. The following cognate Riccatians are interesting, as they give other conditions of solubility of equation (1) besides those in the last article:

$$\frac{dy}{dx}-\frac{ay}{x}+bx^{-n-2}y^2-cx^{m+2n+2}=0\ \ldots\ldots(12),$$

where
$$x^{2n+2}u=y+\frac{n+1-a}{b}x^{n+1}\ldots\ldots\ldots\ldots(13);$$

$$\frac{dy}{dx}-\frac{ay}{x}+bx^{-m-2-2a}y^2-cx^{2a+2m+n+2}=0\ldots\ldots(14),$$

where $$u = \frac{cx^{2a+2m+2}}{by + (a+m+1)\,x^{2a+m+1}} \;\ldots\ldots\ldots\ldots (15).$$

From these it follows that $\phi(-a,\ -n-2,\ m+2n+2)$ and $\phi(-a,\ -m-2-2a,\ 2a+2m+n+2)$ are also conditions of solubility of (1).

In particular cases it is possible that some of the above conditions of solubility may be identical. The following equations are a little more general than the above:

$$\frac{dy}{dx} + \frac{\alpha - a + 2n + 2}{x}\,y + bx^{n+\alpha}y^2 - cx^{m-\alpha} = 0 \;\ldots (16),$$

where $$u = x^{\alpha}y + \frac{n+1-a}{b}\,x^{-n-1} \;\ldots\ldots\ldots\ldots (17);$$

$$\frac{dy}{dx} + \frac{a - \alpha + 2m + 2}{x}\,y + bx^{m-\alpha}y^2 - cx^{n+\alpha} = 0 \;\ldots. (18),$$

where $$u = \frac{cx^{\alpha}}{by + (a+m+1)\,x^{\alpha-m-1}} \;\ldots\ldots\ldots\ldots (19).$$

5. *Particular Cases of Solubility of the Equation* (1).

In (2) put $a + \alpha = 0$ and $m - \alpha = n + \alpha$, then it becomes

$$\frac{dy}{dx} + (by^2 - c)\,x^{\frac{1}{2}(n+m)} = 0 \;\ldots\ldots\ldots\ldots\ldots (20),$$

which is easily integrated by the usual methods. It is seen, therefore, that (1) is soluble when $2a + m - n = 0$.

In (4) put $a - \alpha - 2n - 2 = 0$ and $m - \alpha = n + \alpha$, then it assumes the form (20). Equation (1) is, therefore, soluble when $2a = m + 3n + 4$.

Equation (1) is homogeneous, and, therefore, soluble when $m + n = 2$.

In (18) put $a - \alpha + 2m + 2 = 0$ and $m - \alpha = n + \alpha$, then it assumes the form (20); therefore (1) is soluble when $2a + 3m + n + 4 = 0$.

6. Having regard to a series of transformations similar to that employed in (4), it will conduce to simplicity to make $\alpha = m - n$, then (4) becomes

$$\frac{dy}{dx} + \frac{a - (m+n+2)}{x}\,y + bx^n y^2 - cx^m = 0 \ldots\ldots (21),$$

where $$u = \frac{cx^{m-n}}{by} + \frac{n+1-a}{b}\,x^{-n-1} \;\ldots\ldots\ldots\ldots (22).$$

If i of these cognate Riccatians be taken in accordance with (21) and (22), which are forms well adapted for such

a series of transformations, then $a_i = a - i(m+n+2)$. Apply this equation to the conditions of solubility of (1), given in Art. 5, and we obtain the following values of i which will render (1) soluble:

$$i = \frac{2a+m-n}{2(m+n+2)} = \frac{2a-m-3n-4}{2(m+n+2)} = \frac{2a+3m+n+4}{2(m+n+2)} = \frac{m+3n+4-2a}{2(m+n+2)}.$$

In these equations i must be a positive integer. Boole's results are included in the above by suitable substitutions. Had Boole assumed $y = \frac{cx^n}{by_1} + \frac{a}{b}$ instead of $y = \frac{x^n}{y_1} + \frac{a}{b}$ (see Boole, *Differential Equations*, p. 93), the resulting equation would have been more simple for further transformations.

7. In (18) put $a = m - n$, then it becomes suitable for a series of transformations

$$\frac{dy}{dx} + \frac{a+m+n+2}{x} y + bx^n y^2 - cx^m = 0 \ldots\ldots (23),$$

where
$$u = \frac{cx^{m-n}}{by + (a+m+1)x^{-n-1}} \ldots\ldots\ldots\ldots (24).$$

If i of these cognate Riccatians be considered in accordance with equations (23) and (24), we obtain $a_i = a + i(m+n+2)$. Apply this equation to the conditions of solubility given in Art. 5, and we obtain the following values of i which will make (1) soluble:

$$i = \frac{n-m-2a}{2(m+n+2)} = \frac{m+3n+4-2a}{2(m+n+2)}$$
$$= -\frac{3m+n+4+2a}{2(m+n+2)} = -\frac{m+3n+4+2a}{2(m+n+2)},$$

where i is a positive integer.

8. Equation (1) becomes the ordinary Riccati's equation when $a = 0$ and $n = 0$. Substitute these values in the above conditions, and there results

$$m = -\frac{4i}{2i \pm 1}.$$

This condition renders soluble the equation

$$\frac{du}{dx} + bu^2 = cx^m \ldots\ldots\ldots\ldots\ldots\ldots (25),$$

as is well known.

Havant, 1876.

ON THE RESOLUTION OF THE PRODUCT OF TWO SUMS OF EIGHT SQUARES INTO THE SUM OF EIGHT SQUARES.

By *J. J. Thomson*, Trinity College, Cambridge.

THE product of two sums of eight squares seems to have been first resolved into the sum of eight squares by Mr. J. T. Graves. Prof. Young, in a long paper on the subject in the *Irish Transactions* for 1848, works out the resolution, and also gives a proof that the corresponding proposition does not hold for sixteen squares, though Le Besgue, in his *Théorie des Nombres*, says Genocchi has proved that the product of two sums, each of 2^n squares, can be expressed as the sum of 2^n squares. Brioschi, in *Crelle*, t. 52, deduces the resolution for eight squares from some propositions about determinants. The following proof, however, is shorter than any of those mentioned above:

To prove

$$
\begin{aligned}
&(a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2+a_7^2+a_8^2) \\
&\qquad\qquad \times(\alpha_1^2+\alpha_2^2+\alpha_3^2+\alpha_4^2+\alpha_5^2+\alpha_6^2+\alpha_7^2+\alpha_8^2) \\
&=(a_1\alpha_1+a_2\alpha_2+a_3\alpha_3+a_4\alpha_4+a_5\alpha_5+a_6\alpha_6+a_7\alpha_7+a_8\alpha_8)^2 \\
&\quad+\left(\begin{vmatrix} a_1 & a_2 \\ \alpha_1 & \alpha_2 \end{vmatrix}+\begin{vmatrix} a_3 & a_4 \\ \alpha_3 & \alpha_4 \end{vmatrix}+\begin{vmatrix} a_5 & a_6 \\ \alpha_5 & \alpha_6 \end{vmatrix}+\begin{vmatrix} a_7 & a_8 \\ \alpha_7 & \alpha_8 \end{vmatrix}\right)^2 \\
&\quad+\left(\begin{vmatrix} a_1 & a_3 \\ \alpha_1 & \alpha_3 \end{vmatrix}+\begin{vmatrix} a_4 & a_2 \\ \alpha_4 & \alpha_2 \end{vmatrix}+\begin{vmatrix} a_5 & a_7 \\ \alpha_5 & \alpha_7 \end{vmatrix}+\begin{vmatrix} a_8 & a_6 \\ \alpha_8 & \alpha_6 \end{vmatrix}\right)^2 \\
&\quad+\left(\begin{vmatrix} a_4 & a_1 \\ \alpha_4 & \alpha_1 \end{vmatrix}+\begin{vmatrix} a_3 & a_2 \\ \alpha_3 & \alpha_2 \end{vmatrix}+\begin{vmatrix} a_5 & a_8 \\ \alpha_5 & \alpha_8 \end{vmatrix}+\begin{vmatrix} a_6 & a_7 \\ \alpha_6 & \alpha_7 \end{vmatrix}\right)^2 \\
&\quad+\left(\begin{vmatrix} a_1 & a_5 \\ \alpha_1 & \alpha_5 \end{vmatrix}+\begin{vmatrix} a_6 & a_2 \\ \alpha_6 & \alpha_2 \end{vmatrix}+\begin{vmatrix} a_7 & a_3 \\ \alpha_7 & \alpha_3 \end{vmatrix}+\begin{vmatrix} a_4 & a_8 \\ \alpha_4 & \alpha_8 \end{vmatrix}\right)^2 \\
&\quad+\left(\begin{vmatrix} a_1 & a_6 \\ \alpha_1 & \alpha_6 \end{vmatrix}+\begin{vmatrix} a_2 & a_5 \\ \alpha_2 & \alpha_5 \end{vmatrix}+\begin{vmatrix} a_3 & a_8 \\ \alpha_3 & \alpha_8 \end{vmatrix}+\begin{vmatrix} a_4 & a_7 \\ \alpha_4 & \alpha_7 \end{vmatrix}\right)^2 \\
&\quad+\left(\begin{vmatrix} a_1 & a_7 \\ \alpha_1 & \alpha_7 \end{vmatrix}+\begin{vmatrix} a_8 & a_2 \\ \alpha_8 & \alpha_2 \end{vmatrix}+\begin{vmatrix} a_3 & a_5 \\ \alpha_3 & \alpha_5 \end{vmatrix}+\begin{vmatrix} a_6 & a_4 \\ \alpha_6 & \alpha_4 \end{vmatrix}\right)^2 \\
&\quad+\left(\begin{vmatrix} a_1 & a_8 \\ \alpha_1 & \alpha_8 \end{vmatrix}+\begin{vmatrix} a_2 & a_7 \\ \alpha_2 & \alpha_7 \end{vmatrix}+\begin{vmatrix} a_6 & a_3 \\ \alpha_6 & \alpha_3 \end{vmatrix}+\begin{vmatrix} a_5 & a_4 \\ \alpha_5 & \alpha_4 \end{vmatrix}\right)^2
\end{aligned}
$$

We can see that this would be true if, after expanding the squares, the terms involving the products of the determinants disappeared; for, to take an example,

$$\begin{vmatrix} a_2 a_5 \\ \alpha_2 \alpha_5 \end{vmatrix}^2 = a_2^2\alpha_5^2 + a_5^2\alpha_2^2 - 2a_2a_5\alpha_2\alpha_5,$$

the first two terms occur on the left-hand side of the equation, and the last term is cancelled by a term $+2a_2a_5\alpha_2\alpha_5$ from the first square on the right-hand side of the equation; we thus get all the terms we want from the squares of the determinants, and if the sum of the products of the determinants, two and two, vanishes, the equation will be true.

To prove that the sum does vanish we notice

$$\begin{vmatrix} a_1 a_2 \\ \alpha_1 \alpha_2 \end{vmatrix} \begin{vmatrix} a_3 a_4 \\ \alpha_3 \alpha_4 \end{vmatrix} + \begin{vmatrix} a_1 a_3 \\ \alpha_1 \alpha_3 \end{vmatrix} \begin{vmatrix} a_4 a_2 \\ \alpha_4 \alpha_2 \end{vmatrix} + \begin{vmatrix} a_4 a_1 \\ \alpha_4 \alpha_1 \end{vmatrix} \begin{vmatrix} a_3 a_2 \\ \alpha_3 \alpha_2 \end{vmatrix} = 0,$$

if we denote $\begin{vmatrix} a_1 a_2 \\ \alpha_1 \alpha_2 \end{vmatrix}$ by (1.2).

We may express this

$$(1.2)(3.4) + (1.3)(4.2) + (4.1)(3.2) = 0 \ldots\ldots (\text{I.}).$$

Write, for convenience, the last seven terms on the right-hand side again, using this notation,

$$\begin{aligned} &\{(1.2) + (3.4) + (5.6) + (7.8)\}^2 \\ +&\{(1.3) + (4.2) + (5.7) + (8.6)\}^2 \\ +&\{(4.1) + (3.2) + (5.8) + (6.7)\}^2 \\ +&\{(1.5) + (6.2) + (7.3) + (4.8)\}^2 \\ +&\{(1.6) + (2.5) + (3.8) + (4.7)\}^2 \\ +&\{(1.7) + (8.2) + (3.5) + (6.4)\}^2 \\ +&\{(1.8) + (2.7) + (6.3) + (5.4)\}^2. \end{aligned}$$

We can easily prove that the sum of the products vanish, for suppose we have product (6.2) (7.3) from the fourth row, we have (3.2) (6.7) from the third row, and (2.7) (6.3) from the seventh row, and the sum of these three products vanishes by equation (I.). There will be 42 products, two and two, but in applying this process we take into account 3 each time, so that we have only to go through this process 14 times, and we shall thus find that the sum of the products vanishes.

$$\frac{\pi^3}{48} = \tfrac{1}{3}\,\frac{1}{2} + \tfrac{1}{5}\,\frac{1.3}{2.4}\left(1 + \frac{1}{3^2}\right) + \tfrac{1}{7}\,\frac{1.3.5}{2.4.6}\left(1 + \frac{1}{3^2} + \frac{1}{5^2}\right)$$

$$+ \tfrac{1}{9}\,\frac{1.3.5.7}{2.4.6.8}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2}\right) + \&c. \ldots\ldots \text{(xii)}.$$

$$\frac{\pi^3}{3^4\sqrt{3}} = \frac{1}{2.3} + \frac{1.2}{3.4.5}\left(1 + \frac{1}{2^2}\right) + \frac{1.2.3}{4.5.6.7}\left(1 + \frac{1}{2^2} + \frac{1}{3^2}\right)$$

$$+ \frac{1.2.3.4}{5.6.7.8.9}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2}\right) + \&c. \ldots\ldots\ldots \text{(xiii)}.$$

$$\frac{\pi^4}{48} = \tfrac{1}{2}\,\frac{2}{3} + \tfrac{1}{3}\,\frac{2.4}{3.5}\left(1 + \frac{1}{2^2}\right) + \tfrac{1}{4}\,\frac{2.4.6}{3.5.7}\left(1 + \frac{1}{2^2} + \frac{1}{3^2}\right) + \&c. \ldots \text{(xiv)}.$$

$$\frac{\pi^4}{1944} = \frac{1}{2.3.4} + \frac{1.2}{3.4.5.6}\left(1 + \frac{1}{2^2}\right) + \frac{1.2.3}{4.5.6.7.8}\left(1 + \frac{1}{2^2} + \frac{1}{3^2}\right) + \&c.$$

$$\ldots\ldots\ldots\ldots\ldots\ldots \text{(xv)}.$$

§ 2. Of these (i) is well known, and is obtained at once from the series for arc sin x by putting $x = 1$; it is included for the sake of completeness, as it somewhat resembles in form several of the others. (ii) is obtained by putting $x = 2^{-\frac{1}{2}}$ in the formula

$$\frac{\text{arc}\sin x}{\sqrt{(1 - x^2)}} = x + \frac{2}{3}\,x^3 + \frac{2.4}{3.5}\,x^5 + \frac{2.4.6}{3.5.7}\,x^7 + \&c. \ldots \text{(xvi)}.$$

Differentiate this equation, and we have

$$\frac{x \text{ arc}\sin x}{(1 - x^2)^{\frac{3}{2}}} + \frac{1}{1 - x^2} = 1 + \frac{2}{1}\,x^2 + \frac{2.4}{1.3}\,x^4 + \frac{2.4.6}{1.3.5}\,x^6 + \&c. \ldots \text{(xvii)},$$

which gives (iii) on taking $x = 2^{-\frac{1}{2}}$. If in (xvi) we replace x by $\frac{1}{2}x$, we find, after transforming the coefficients as explained in § 3, that

$$4\,\frac{\text{arc}\sin \frac{1}{2}x}{\sqrt{(4 - x^2)}} = x + \frac{1}{2.3}\,x^3 + \frac{1.2}{3.4.5}\,x^5 + \frac{1.2.3}{4.5.6.7}\,x^7 + \&c. \ldots \text{(xviii)},$$

which gives (iv) when $x = 1$. Differentiating (xviii),

$$4\left\{\frac{x \text{ arc}\sin \frac{1}{2}x}{(4 - x^2)^{\frac{3}{2}}} + \frac{1}{4 - x^2}\right\} = 1 + \frac{1}{2}\,x^2 + \frac{1.2}{3.4}\,x^4 + \frac{1.2.3}{4.5.6}\,x^6 + \&c.$$

$$\ldots\ldots\ldots\ldots\ldots \text{(xix)},$$

from which (v) is derived by putting $x=1$. (vi) is obtained by subtracting (iv) from (v). (vii) and (viii) follow from the equation

$$(\text{arc}\sin x)^2 = x^2 + \tfrac{1}{2}\,\frac{2}{3}\,x^4 + \tfrac{1}{3}\,\frac{2.4}{3.5}\,x^6 + \tfrac{1}{4}\,\frac{2.4.6}{3.5.7}\,x^8 + \&\text{c.} \ldots \text{(xx)},$$

by putting $x=2^{-\frac{1}{2}}$ and $x=1$. (ix) is obtained from

$$2\,(\text{arc}\sin\tfrac{1}{2}x)^2 = \frac{1}{2}\,x^2 + \frac{1}{2.3.4}\,x^4 + \frac{1.2}{3.4.5.6}\,x^6 + \&\text{c.}\ldots\ldots\text{(xxi)},$$

and (x) from

$$2\,\{x\,(\text{arc}\sin\tfrac{1}{2}x)^2 + 2\ \text{arc}\sin\tfrac{1}{2}x\,\sqrt{(4-x^2)} - 2x\} - \tfrac{1}{6}x^3$$

$$= \frac{1}{2.3.4.5}\,x^5 + \frac{1.2}{3.4.5.6.7}\,x^7 + \frac{1.2.3}{4.5.6.7.8.9}\,x^9 + \&\text{c.}\ldots\text{(xxii)},$$

which results from the integration of (xxi). The integration of (xx) gives

$$x\,(\text{arc}\sin x)^2 + 2\ \text{arc}\sin x\,\sqrt{(1-x^2)} - 2x - \tfrac{1}{3}x^3 = \tfrac{1}{2}\,\frac{2}{3.5}\,x^5 + \&\text{c.}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\text{(xxiii)},$$

from which (xi) is derived.

(xii) and (xiii) are obtained by equating the coefficients of x^3 in

$$\sin\tfrac{1}{2}\pi x = \quad x - \frac{x(x^2-1^2)}{3\,!} + \frac{x(x^2-1^2)(x^2-3^2)}{5\,!} - \&\text{c.} \ldots\text{(xxiv)},$$

$$\sin\tfrac{1}{3}\pi x = \tfrac{1}{2}\sqrt{3}\left\{x - \frac{x(x^2-1^2)}{3\,!} + \frac{x(x^2-1^2)(x^2-2^2)}{5\,!} - \&\text{c.}\right\}\ldots\text{(xxv)},$$

and (xiv) and (xv) by equating the coefficients of x^4 in

$$\cos\tfrac{1}{2}\pi x = 1 - \frac{x^2}{2\,!} + \frac{x^2\,(x^2-2^2)}{4\,!} - \frac{x^2\,(x^2-2^2)\,(x^2-4^2)}{6\,!} + \&\text{c.}$$

$$\ldots\ldots\ldots\ldots\ldots\text{(xxvi)},$$

$$\cos\tfrac{1}{3}\pi x = 1 - \frac{x^2}{2\,!} + \frac{x^2\,(x^2-1^2)}{4\,!} - \frac{x^2\,(x^2-1^2)\,(x^2-2^2)}{6\,!} + \&\text{c.}$$

$$\ldots\ldots\ldots\ldots\ldots\text{(xxvii)}.$$

§ 3. The equations (iv) and (v) were given in vol. II., p. 143, of the *Messenger* (January, 1873); they were there derived from (xxv) and (xxvii) by equating the coefficients

of x and x^2. There is an error in (iv) as it appears in vol. II., p. 141, which was corrected in vol. II., p. 153.

The series for π, π^2; π^3, &c., alluded to in vol II., p. 142, are included in those given in § 1. They were obtained in the first instance by equating coefficients in (xxiv)...(xxvii), and in the other similar formulæ which occur in vol. II., pp. 138–142 and 153–157; but the method of § 2 is rather preferable, as the formulæ for arc sin$x$ and (arc sin$x$)^2 are better known than (xxiv)...(xxvii). The peculiar form of the terms in (iv) and (ix) suggested the determination of the values of the series in (v) and (x).

The coefficients in (xvi) and (xviii) appear to be very dissimilar in form, although they only differ by powers of 2; the transformation employed is

$$(2^2)\,3.5 = 3.4.5\ ,\quad (2^4)\,3.5.7.9 = 5.6.7.8.9,$$

$$(2^3)\,3.5.7 = 4.5.6.7,\quad (2^5)\,3.5.7.9.11 = 6.7.8.9.10.11,\ \&c.$$

The proof is very simple, viz. $3.5.7\ldots(2n+1)$

$$= \frac{2.3.4\ldots(2n+1)}{2.4.6\ldots2n} = \frac{(2n+1)\,!}{2^n.\,n\,!} = \frac{1}{2^n}\,(n+1)\,(n+2)\ldots(2n+1).$$

We also have, on multiplying by 2,

$$(2^3)\,3.5=4.5.6\,;\ 2^4\,(3.5.7) = 5.6.7.8\,;\ (2^5)\,3.5.7.9 = 6.7.8.9.10,\ \&c.$$

§ 4. The most curious of the series is perhaps (v); it may be written

$$\frac{2\pi}{9\sqrt{3}} + \tfrac{1}{3} = \frac{1}{C_{2,1}} + \frac{1}{C_{4,2}} + \frac{1}{C_{6,3}} + \&c.$$

where $C_{n,r}$ denotes the number of combinations of n things r together, and also in the form

$$\frac{4\sqrt{\pi}}{9\sqrt{3}} - \tfrac{1}{3} = \frac{1}{3} + \frac{1.2}{4.5} + \frac{1.2.3}{5.6.7} + \frac{1.2.3.4}{6.7.8.9} + \&c\ldots\ \text{(xxviii)}.$$

In connexion with (xix) it may be mentioned that the series with reciprocal coefficients, viz.

$$1 + \frac{2}{1}x^2 + \frac{3.4}{1.2}x^4 + \frac{4.5.6}{1.2.3}x^6 + \&c.,$$

is the development of $(1-4x^2)^{-\frac{1}{2}}$, and is convergent if $x < \tfrac{1}{2}$.

§ 5. If we equate the coefficients of x^6 in (xxvii), we have

$$\frac{\pi^6}{524,880} = \frac{1.2}{3.4.5.6}\left(\frac{1}{1^2 2^2}\right) + \frac{1.2.3}{4.5.6.7.8}\left(\frac{1}{1^2 2^2} + \frac{1}{1^2 3^2} + \frac{1}{2^2 3^2}\right)$$

$$+ \frac{1.2.3.4}{5.6.7.8.9.10}\left(\frac{1}{1^2 2^2} + \frac{1}{1^2 3^2} + \frac{1}{1^2 4^2} + \frac{1}{2^2 3^2} + \frac{1}{2^2 4^2} + \frac{1}{3^2 4^2}\right) + \&c. \qquad \text{.................(xxix)},$$

and similar results follow from equating coefficients in (xxiv), &c. If we equate the coefficients of x^8, we have series in which the sums of the combinations of $1^{-2}, 2^{-2}, 3^{-2}, \dots$ three together are involved, and so on.

§ 6. By converting (ii.), (iii.), (v.), and (iv.) into continued fractions, we find

$$1 - \frac{2}{\pi} = \frac{1.1}{4-}\,\frac{2.3}{7-}\,\frac{3.5}{10-}\,\frac{4.7}{13-}\,\frac{5.9}{16-\&c.},$$

$$\frac{\pi}{\pi+2} = \frac{2.1}{5-}\,\frac{3.3}{8-}\,\frac{4.5}{11-}\,\frac{5.7}{14-}\,\frac{6.9}{17-\&c.},$$

$$\frac{4\pi - 3\sqrt{3}}{2\pi + 3\sqrt{3}} = \frac{1.4}{8-}\,\frac{3.6}{13-}\,\frac{5.8}{18-}\,\frac{7.10}{23-\&c.},$$

$$\frac{\pi}{3\sqrt{3}} = \frac{1}{2-}\,\frac{1.2}{7-}\,\frac{3.4}{12-}\,\frac{5.6}{17-}\,\frac{7.8}{22-\&c.},$$

and with this last result may be compared the two equations*

$$1 = \frac{1}{3-}\,\frac{2.3}{7-}\,\frac{4.5}{11-}\,\frac{6.7}{15-}\,\frac{8.9}{19-\&c.},$$

$$1 = \frac{1.2}{5-}\,\frac{3.4}{9-}\,\frac{5.6}{13-}\,\frac{7.8}{17-\&c.}.$$

§ 7. II. *Products.* If

$$\Pi_p = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right)\left(1 - \frac{1}{7^2}\right)\dots,$$

$$\Pi_c = \left(1 - \frac{1}{4^2}\right)\left(1 - \frac{1}{6^2}\right)\left(1 - \frac{1}{8^2}\right)\left(1 - \frac{1}{9^2}\right)\dots,$$

$$P_p = \left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right)\left(1 - \frac{1}{7^2}\right)\left(1 - \frac{1}{11^2}\right)\dots,$$

$$P_c = \left(1 - \frac{1}{9^2}\right)\left(1 - \frac{1}{15^2}\right)\left(1 - \frac{1}{21^2}\right)\left(1 - \frac{1}{25^2}\right)\dots,$$

* *Proceedings of the London Mathematical Society*, t. v. p. 83 (1874).

in which respectively all prime numbers, all composite numbers, all uneven prime numbers, all uneven composite numbers are involved, so that $\Pi_p = \frac{3}{4}P_p$, then

$$\Pi_p = \frac{6}{\pi^2}, \quad P_p = \frac{8}{\pi^2},$$

$$\Pi_c = \tfrac{1}{12}\pi^2, \quad P_c = \tfrac{1}{32}\pi^3,$$

so that
$$\frac{\Pi_c}{\Pi_p} = \frac{\pi^4}{72}, \quad \frac{P_c}{P_p} = \frac{\pi^5}{256}.$$

§ 8. These results are readily proved, for

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \&c.$$

$$= \frac{1}{\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right)\cdots} = \frac{1}{\Pi_p}.$$

Also
$$\tfrac{1}{4}\pi = \left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right)\left(1 - \frac{1}{7^2}\right)\left(1 - \frac{1}{9^2}\right)\cdots,$$

as is evident from Wallis's formula, or by putting $x = 1 + \epsilon$ (ϵ infinitesimal) in

$$\cos \tfrac{1}{2}\pi x = (1 - x^2)\left(1 - \frac{x^2}{3^2}\right)\left(1 - \frac{x^2}{5^2}\right)\cdots,$$

and
$$\tfrac{1}{2} = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\left(1 - \frac{1}{5^2}\right)\cdots,$$

as is evident by putting $x = 1 + \epsilon$ in

$$\frac{\sin \pi x}{\pi x} = (1 - x^2)\left(1 - \frac{x^2}{2^2}\right)\left(1 - \frac{x^2}{3^2}\right)\cdots.$$

Thus
$$P_p P_c = \tfrac{1}{4}\pi, \text{ and } \Pi_p \Pi_c = \tfrac{1}{2},$$

and these, in conjunction with

$$\Pi_p = \tfrac{3}{4}P_p, \quad \Pi_p = \frac{6}{\pi^2}$$

give Π_p, P_p, Π_c, P_c as above.

ON AN ARITHMETICAL THEOREM OF PROFESSOR SMITH'S.

By Professor *Paul Mansion.*

I. WE may write the theorem* thus:

$$\begin{vmatrix} 1, & 1, & 1, & 1, & 1, & 1, & \ldots 1 \\ 1, & 2, & 1, & 2, & 1, & 2, & \ldots \\ 1, & 1, & 3, & 1, & 1, & 3, & \ldots \\ 1, & 2, & 1, & 4, & 1, & 2, & \ldots \\ 1, & 1, & 1, & 1, & 5, & 1, & \ldots \\ 1, & 2, & 3, & 2, & 1, & 6, & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1, & \ldots & \ldots & \ldots & \ldots & \ldots & L \end{vmatrix} = \phi(1)\,\phi(2)\,\phi(3)\,\phi(4)\ldots\phi(L).$$

The determinant is formed according to the three following laws: 1°. The principal diagonal consists of the series of natural numbers. 2°. The determinant is symmetrical. 3°. The $m-1$ first elements of the m^{th} line (or column) are the same as the elements situated in the same columns (or lines) on the parallel to the second diagonal which joins the $(m-1)^{\text{th}}$ element of the first line and of the first column. If $L = a^\alpha b^\beta \ldots l^\lambda$, we suppose that

$$\phi(L) = (a^\alpha - a^{\alpha-1})(b^\beta - b^{\beta-1})\ldots(l^\lambda - l^{\lambda-1}) = l_1 - l_2 + l_3 - l_4 + \&c.$$

II. Suppose that 1, 2, 3... represent, not the natural numbers, but any algebraical symbols x_1, x_2, x_3... and that

$$\phi(L) = l_1 - l_2 + l_3 - l_4 + \&c. = x_{l_1} - x_{l_2} + x_{l_3} - x_{l_4} + \&c.,$$

then Professor Smith's theorem is still true.

III. In the determinant, when 1, 2, 3...L have an arithmetical signification, we may write instead of them

$$\Sigma\phi(1),\ \Sigma\phi(2),\ \Sigma\phi(3), \ldots,\ \Sigma\phi(L) = \phi(L_1) + \phi(L_2) + \&c.,$$

or, putting for shortness $(L) = \phi(L)$,

$$\Sigma(1),\ \Sigma(2),\ \Sigma(3),\ \ldots,\ \Sigma(L) = (L_1) + (L_2) + \&c.$$

* Professor H. J. S. Smith's theorem was published in the *Proceedings of the London Mathematical Society*, vol. VII., pp. 208–212 (On the Value of a Certain Arithmetical Determinant); the theorem is also stated in vol. VI. p. 31 of the *Messenger* (June, 1876).

Thus

$$\begin{vmatrix} (1), & (1) & , & (1) & , \ldots\ldots\ldots\ldots (1) \\ (1), & (1)+(2), & & (1) & , \ldots\ldots\ldots\ldots \\ (1), & (1) & , & (1)+(3), & \ldots\ldots\ldots\ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ (1) & \ldots\ldots & & , & (L_1)+(L_2)+\&c. \end{vmatrix} = (1)\,(2)\,(3)\ldots(L).$$

IV. Finally, instead of (1), (2), (3)..., we may write any algebraical symbols; which gives the solution of this question: Express any product (1) (2) (3)...(L) as a determinant formed according to the above laws, except that the terms of the diagonal are not the series of natural numbers.

Example.

$$x^5 = \begin{vmatrix} x, & x, & x, & x, & x \\ x, & 2x, & x, & x, & x \\ x, & x, & 2x, & x, & x \\ x, & 2x, & x, & 3x, & x \\ x, & x, & x, & x, & x \end{vmatrix}.$$

We may call such determinants Smithian determinants.

ON THE DEVELOPMENT OF $\left(\frac{z}{1-e^{-z}}\right)^a$ IN A SERIES.

By *M. Édouard Lucas.*

MM. Laurent and Le Paige* have recently given the development of $u^a = \left(\frac{z}{1-e^{-z}}\right)^a$ in a series of ascending powers of z. This can also be performed in the following simple manner:

Let $B_{a,p}$ be the coefficient of $\frac{z^p}{1.2.3\ldots p}$ in the development of u^a; we shall call $B_{a,p}$ the p^{th} Bernoullian number of order a.

We have $$u = e^{B_1 z}, \quad u^a = e^{B_a z},$$

wherein we are to replace powers of B_1 and B_a by *second suffixes* and $B_{1,n}$ by the n^{th} Bernoullian number, so that

$$B_{1,0} = 1, \quad B_{1,1} = \tfrac{1}{2}, \quad B_{1,2} = \tfrac{1}{6}, \quad B_{1,4} = -\tfrac{1}{30} \ldots .$$

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* Laurent, *Nouvelles Annales de Mathématiques*, 1875, t. XIV. p. 355.
Le Paige, *Annales de la Société scientifique de Bruxelles*, 1876.

We have then, by definition,

$$B_{\alpha,p} = [B_1' + B_1'' + \ldots + B_1^{(\alpha)}]^p,$$

on replacing $[B_1^{(\omega)}]^n$ by $B_{1,n}$, and, more generally,

$$B_{\lambda+\mu+\nu+\ldots,p} = [B_\lambda + B_\mu + B_\nu + \ldots]^p.$$

When α is a negative integer, we obtain, on changing the sign,

$$u^{-\alpha} = \left(\frac{1-e^{-z}}{z}\right)^{\alpha};$$

whence, denoting by $C_{\alpha,p}$ the number of combinations of α things taken p together,

$$B_{-\alpha,p} = \frac{(-1)^p}{(p+1)(p+2)\ldots(p+\alpha)}$$
$$\times [\alpha^{p+\alpha} - C_{\alpha,1}(\alpha-1)^{p+\alpha} + C_{\alpha,2}(\alpha-2)^{p+\alpha}\ldots \pm C_{\alpha,\alpha-1}].$$

But

$$\Delta^\alpha x^m = (x+\alpha)^m - C_{\alpha,1}(x+\alpha-1)^m + C_{\alpha,2}(x+\alpha-2)^m + \ldots + (-1)^\alpha x^m,$$

so that $$B_{-\alpha,p} = \frac{(-1)^p \Delta^\alpha 0^{p+\alpha}}{(p+1)(p+2)\ldots(p+\alpha)} \quad \ldots\ldots\ldots\ldots(1).$$

As a particular case,

$$B_{-1,p} = \frac{(-1)^p}{p+1}.$$

By differentiation,

$$z\frac{d(u^\alpha)}{dz} = \alpha(z+1)u^\alpha - \alpha u^{\alpha+1},$$

and, equating the coefficients of $\dfrac{z^{p-1}}{1.2.3\ldots(p-1)}$,

$$\alpha B_{\alpha+1,p} = (\alpha-p)B_{\alpha,p} + \alpha p B_{\alpha,p-1} \quad \ldots\ldots\ldots\ldots(2).$$

Thus, for $\alpha = 1$,

$$B_{2,p} = pB_{1,p-1} - (p-1)B_{1,p}.$$

Put $$\frac{B_{1,p}}{p} = B_p,$$

and we have the symbolical formula

$$1.2.3\ldots B_{\alpha+1,p} = p(p-1)\ldots(p-\alpha)B^{p-\alpha}(1-B)(2-B)\ldots(\alpha-B)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

in which the powers of B are to be replaced by suffixes.

Replace p by $p-1$, and we obtain

$$1.2.3\ldots\alpha B_{\alpha+1,p-1}$$
$$=(p-1)(p-2)\ldots(p-\alpha-1)B^{p-\alpha-1}(1-B)\ldots(\alpha-B)\ldots\ldots(4).$$

Multiply (3) by $\alpha-p+1$, (4) by $p(\alpha+1)$, and add, taking account of the formula (2); we thus find

$$1.2.3\ldots\alpha(\alpha+1)B_{\alpha+2,p}$$
$$=(p-1)(p-2)\ldots(p-\alpha-1)B^{p-2-1}(1-B)(2-B)\ldots(\alpha-B)$$
$$\times[p(\alpha-p+1)B+p(\alpha+1)(p-\alpha-1)].$$

Simplifying this, we reproduce the formula (3), α being changed into $\alpha+1$. This formula, true for $\alpha=1, 2\ldots$ is thus generally true: it expresses the Bernoullian numbers of order α as a linear function of α consecutive Bernoullian numbers of the first order.

Paris, *September*, 1877.

ON THE SUCCESSIVE SUMMATIONS OF $1^m+2^m+3^m\ldots+x^m$.

By *M. Édouard Lucas.*

LET $\quad S_{1,m}(x)=1^m+2^m+3^m+\ldots+x^m,$

$$S_{p,m}(x)=S_{p-1,m}(1)+S_{p-1,m}(2)+S_{p-1,m}(3)+\ldots+S_{p-1,m}(x),$$

and suppose that

$$S_{0,0}=1,\quad S_{p,0}=\frac{x(x+1)(x+2)\ldots(x+p-1)}{1.2.3\ldots p}.$$

We have the symbolical formula

$$S_{1,m}=\frac{(x+B_1)^{m+1}-B_1^{\,m+1}}{m+1}\ \ldots\ldots\ldots\ldots\ldots(1),$$

in the development of which the $m+1$ powers of B_1 are to be replaced by second suffixes and $B_{1,n}$ by the n^{th} Bernoullian number, with its proper sign. Differentiating the two sides of the equation, we have, as a formula to calculate the Bernoullian numbers, the identity

$$(x+1)^m=\frac{(x+1+B_1)^{m+1}-(x+B_1)^{m+1}}{m+1},$$

or, more generally,

$$f'(x+1)=f(x+1+B_1)-f(x+B_1).$$

To calculate $S_{2,m}$, we form the table

$$\begin{array}{l} 1^m, \\ 1^m+2^m, \\ 1^m+2^m+3^m, \\ \cdots\cdots\cdots\cdots\cdots \\ 1^m+2^m+3^m+\ldots+x^m, \end{array}$$

and add the columns; the sum of the p^{th} column is

$$(x-p+1)\,p^m \text{ or } (x+1)\,p^m-p^{m+1}.$$

Thus $$S_{2,m}=(x+1)\,S_{1,m}-S_{1,m+1}\ \ldots\ldots\ldots\ldots\ldots(2);$$

or, symbolically expressed,

$$S_{2,m}=S_1^m\,(x+1-S_1).$$

For example $$S_{2,1}=\frac{x\,(x+1)\,(x+2)}{1.2.3},$$

$$S_{2,2}=\frac{x\,(x+1)^2\,(x+2)}{12},\quad S_{2,3}=\frac{x\,(x+1)\,(x+2)\,(x^2+6x+3)}{60}.$$

In general

$$S_{p+1,m}=\frac{S_1^m\,(x+1-S_1)\,(x+2-S_1)\ldots(x+p-S_1)}{1.2.3\ldots p}\ldots\ldots(3).$$

In fact, changing x into $x+1$ the first side of the formula (3) is increased by

$$S_{p,m}\,(x+1),$$

and the second by

$$S^m\,(x+2-S_1)\,(x+3-S_1)\ldots(x+p-S_1),$$

that is, by the second side of (3), when x is replaced by $x+1$ and p by $p-1$.

The formula (1) gives by summation

$$S_{2,m}=\frac{(S_1+B_1)^{m+1}-S_1^{\,0}B_1^{\,m+1}}{m+1},$$

and generally, after p summations,

$$S_{p+1,m}=\frac{(S_p+B_1)^{m+1}-S_p^{\,0}B_1^{\,m+1}}{m+1}.$$

Changing p into $p+1$, we deduce

$$S_{p+1,m}=\frac{(S_p+B_1+B_1)^{m+2}-S_p^{\,0}\,(B_1+B_1)^{m+2}}{(m+1)\,(m+2)}-\frac{S^0_{p+1}B_1^{\,m+1}}{m+1},$$

and putting

$$B_{a,p} = [B_1 + B_1 + \dots B_1^{(\infty)}]^p,$$

we have

$$S_{p+2,m} = \frac{(S_p + B_2)^{m+2} - S_p^0 B_2^{m+2}}{(m+1)(m+2)} - \frac{S^0_{p+1} B_1^{m+1}}{m+1}.$$

In general we should find in the same manner

$$S_{p+q,m} = \frac{(S_p + B_q)^{m+q} - S_p^0 B_q^{m+q}}{(m+1)(m+2)\dots(m+q)} - \frac{S^0_{p+1} B_{q-1}^{\ m+q-1}}{(m+1)p\dots(m+q-1)}$$

$$- \frac{S^0_{p+2} B_{q-2}^{\ m+q-2}}{(m+1)\dots(m+q-2)} - \dots - \frac{S^0_{p+q-1} B_1^{m+1}}{m+1} \dots\dots(4).$$

Putting $p = 0$, we obtain the development of $S_{q,m}$ as a function S_0 or of x.

Paris, *September*, 1877.

CUBE ROOTS OF PRIMES TO 31 PLACES.

By *S. M. Drach, F.R.A.S.*

THE 56-place values of the cube roots of 2 and 4 given on p. 54, have reminded me that twelve years ago I calculated the values of the cube roots of the primes from 2 to 127 to 33 places. The extraction of cube roots to a number of decimal places is so troublesome that it seems desirable to publish these values, which are given in Table I. They were obtained by the usual process of extracting cube roots, and were verified by actual multiplication, the multiplication being contracted throughout to 33 places.

The quantity ε, when preceded by 31 ciphers, is the amount by which the cube of the quantity in the second column differed from the first column, that is to say, for example, by cubing the quantity

1·25992 10498 94873 16476 72106 07278 399,

retaining 33 places throughout the process, I obtained as the cube

2 − ·00000 00000 00000 00000 00000 00000 003,

and in general, cube of number in second column = first column, + ε preceded by 31 ciphers.

The product of the root by itself gave me the root of the square of the number, and Table II. contains the cube roots of the squares of the primes from 2^2 to 127^2 found in this manner.

All the cube roots ought to be accurate to 31 places at least.

I may mention that they occur in my large MS. table of Binary Squares, a copy of which is in the library of the Royal Society, and another in that of the French Institute.

Table I.

N	$\sqrt[3]{N}$	ε
2	1·25992 10498 94873 16476 72106 07278 399	−03
3	1·44224 95703 07408 38232 16383 10780 108	+00
5	1·70997 59466 76696 98935 31088 72543 830	−02
7	1·91293 11827 72389 10119 91168 39548 756	+03
11	2·22398 00905 69315 52116 53633 76722 157	−05
13	2·35133 46877 20757 48950 00163 39956 921	−00
17	2·57128 15906 58235 35545 31872 08741 445	−09
19	2·66840 16487 21944 86733 96273 71970 829	−00
23	2·84386 69798 51565 47769 54394 00958 652	−06
29	3·07231 68256 85847 29331 26380 36360 183	−08
31	3·14138 06523 91393 00449 30758 96462 750	+17
37	3·33222 18516 45953 26009 54505 05135 051	−04
41	3·44821 72403 82730 38410 86376 34932 233	−14
43	3·50339 80603 86724 17061 43337 58189 130	−19
47	3·60882 60801 38694 68925 25172 93399 702	+00
53	3·75628 57542 21072 00661 21096 32059 320	+09
59	3·89299 64158 73260 54646 14847 57149 833	−01
61	3·93649 71831 02173 19582 88513 73032 163	−16
67	4·06154 81004 45679 78908 20615 85799 224	−02
71	4·14081 77494 22853 25000 45188 09325 572	+05
73	4·17933 91963 81231 89205 63766 71392 658	+09
79	4·29084 04270 26207 11162 83142 33454 271	+09
83	4·36207 06714 54837 56471 39794 76679 005	+06
89	4·46474 50955 84537 63343 39684 80965 127	−09
97	4·59470 08922 07039 80609 42964 64422 309	−02
101	4·65700 95078 03835 63042 90105 64845 024	+28
103	4·68754 81476 53597 85820 73434 67619 767	+20
107	4·74745 93985 23400 36029 37741 28935 120	−18
109	4·77685 61810 35016 96494 37334 65132 732	+22
113	4·83458 81271 11639 19899 42141 29566 902	−05
127	5·02652 56953 13479 18113 74068 71623 742	−31

Table II.

N	N^2	$\sqrt[3]{N^2}$
2	4	1·58740 10519 68199 47475 17056 39272 087
3	9	2·08008 38230 51904 11453 00568 24357 885
5	25	2·92401 77382 12866 06550 67873 60137 984
7	49	3·65930 57100 22971 51723 80733 10119 419
11	121	4·94608 74432 48700 86832 36036 57530 282
13	169	5·52877 48136 78872 14147 23447 73085 347
17	289	6·61148 90184 57945 00316 10465 36489 311
19	361	7·12036 73589 01993 65206 96105 01623 781
23	529	8·08757 93990 90064 32667 40060 25366 363
29	841	9·43913 06773 92360 98272 06416 58182 980
31	961	9·86827 24032 18973 92743 85838 62197 395
37	1369	11·10370 24685 86785 33744 67486 52528 462
41	1681	11·89020 21368 72692 61765 69351 98292 247
43	1849	12·27379 79695 21461 01832 02873 77401 952
47	2209	13·02362 56766 89216 42348 31964 44970 526
53	2809	14·10968 26673 64167 77413 93952 61688 715
59	3481	15·15542 10940 02052 57923 35861 72329 105
61	3721	15·49601 00725 71344 48412 72009 22079 464
67	4489	16·49617 29722 33909 80130 79264 30167 879
71	5041	17·14637 16339 35343 48686 08965 02758 734
73	5329	17·46687 61184 08521 19461 82458 41194 894
79	6241	18·41131 15702 02443 39710 00318 34230 104
83	6889	19·02766 05427 66457 44342 27059 68069 770
89	7921	19·93394 87685 46182 08877 61218 08447 890
97	9409	21·11127 62888 48167 62752 48074 83694 822
101	10201	21·68773 75557 75323 39559 26330 98920 636
103	10609	21·97310 76365 70676 46769 31933 87120 413
107	11449	22·53837 07406 28166 32306 04759 95760 252
109	11881	22·81835 49742 92446 77186 38932 28105 547
113	12769	23·37324 23588 08827 22094 28992 55751 637
127	16129	25·26596 05656 46655 34276 77391 28779 193

23, Upper Barnsbury Street, London,
September 3, 1877.

ON CERTAIN PARTIAL DIFFERENTIAL EQUATIONS OF THE SECOND ORDER WHICH HAVE A GENERAL FIRST INTEGRAL.

By *H. W. Lloyd Tanner, M.A.*

1. ADOPTING the usual notation for partial differential coefficients, and representing by V, T, functions of x, y, z, p, q, it is proposed to find under what conditions the equation

$$s + Tt + V = 0 \quad\text{.....................}(1),$$

has a first integral of the form

$$f(p, q, z, x, y) = \phi(x) \quad\text{..................}(2),$$

where ϕ is arbitrary.

This problem was discussed in a former paper,* but the results there obtained were incomplete; and, as the question is not without interest, I venture to submit the present paper, in which it is treated somewhat differently.

2. If we differentiate (2) with respect to y, $\phi(x)$ is eliminated, and we obtain the equation

$$\frac{df}{dp}s + \frac{df}{dq}t + \frac{df}{dz}q + \frac{df}{dy} = 0,$$

which should be the same as (1). On comparing this with (1), we get

$$1 = \mu\frac{df}{dp} \quad\text{..................................}(3, a),$$

$$T = \mu\frac{df}{dq} \quad\text{..................................}(3, b),$$

$$V = \mu\left(\frac{df}{dy} + q\frac{df}{dz}\right) = \mu\left(\frac{df}{dy}\right) \quad\text{...........}(3, c).$$

If from this system we eliminate μ, f, we shall get the relations between T, V which we seek. The only difficulty in the way of performing this elimination arises from (3, c),

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* *Messenger*, vol v. p. 133.

and this difficulty would be overcome if we could separate (3, c) into two equations

$$\Sigma = \mu \frac{df}{dq},$$

$$V - q\Sigma = \mu \frac{df}{dy},$$

Σ being some function of V, T.

Now supposing this separation to be accomplished, the expression

$$dp + Tdq + \Sigma dz + (V - q\Sigma)\, dy, \ = \mu df \ldots\ldots\ldots(4),$$

becomes an exact differential on being divided by μ. To ensure this three conditions are necessary and sufficient (Boole, *Differential Equations*, Chap. XII. Art. 9), viz. these conditions are

$$\left.\begin{array}{l} \dfrac{d}{dq}(V - q\Sigma) - \dfrac{dT}{dy} - T\dfrac{d}{dp}(V - q\Sigma) + (V - q\Sigma)\dfrac{dT}{dp} = 0 \\ \dfrac{d\Sigma}{dy} - \dfrac{d}{dz}(V - q\Sigma) - (V - q\Sigma)\dfrac{d\Sigma}{dp} + \Sigma\dfrac{d}{dp}(V - q\Sigma) = 0 \\ \dfrac{dT}{dz} - \dfrac{d\Sigma}{dq} - \Sigma\dfrac{dT}{dp} + T\dfrac{d\Sigma}{dp} = 0 \end{array}\right\}\ldots(5).$$

The equations (5) constitute a complete solution of our problem. For some applications, however, it is convenient to transform them slightly. From the first of (5) subtract the third multiplied by q. Some easy reductions then give

$$\Sigma = \frac{dV}{dq} - \left(\frac{dT}{dy}\right) + V\frac{dT}{dp} - T\frac{dV}{dp} \ldots\ldots\ldots\ldots(6),$$

which fixes the form of Σ. It is not difficult to verify by means of (3) that the equation

$$\Sigma = \mu \frac{df}{dz}$$

is identically satisfied when Σ is given by (6).

The last two equations of (5) may be written

$$\left(\frac{d\Sigma}{dy}\right) - \frac{dV}{dz} + \Sigma\frac{dV}{dp} - V\frac{d\Sigma}{dp} = 0 \ldots\ldots\ldots\ldots(7),$$

$$\frac{dT}{dz} - \frac{d\Sigma}{dq} + T\frac{d\Sigma}{dp} - \Sigma\frac{dT}{dp} = 0 \ .\ \ldots\ldots\ldots(8).$$

The system (6), (7), (8) is obviously equivalent to the system (5).

3. It may be well to illustrate the above results by applying them to particular cases of a more or less simple character.

Case I. $\Sigma = 0$. Here (6), (7), (8) become

$$\frac{dV}{dq} - \frac{dT}{dy} + V\frac{dT}{dp} - T\frac{dV}{dp} = 0,$$

$$\frac{dV}{dz} = 0, \quad \frac{dT}{dz} = 0,$$

and the first integral is

$$\phi(x) = \int \frac{1}{\mu}(dp + Tdq + Vdy).$$

Case II. $V - q\Sigma = 0$. The system (5) reduce to

$$\frac{dT}{dy} = 0, \quad \frac{d\Sigma}{dy} = 0,$$

$$\frac{dT}{dz} - \frac{d\Sigma}{dq} + T\frac{d\Sigma}{dp} - \Sigma\frac{dT}{dp} = 0,$$

and (4) becomes

$$dp + Tdq + \Sigma dz = \mu df,$$

or

$$dp + Tdq + \frac{V}{q}dz = \mu df.$$

Case III. $V = 0$. In this case (6) becomes

$$\Sigma = -\left(\frac{dT}{dy}\right).$$

Then (7), (8) take the forms

$$\frac{d^2T}{dy^2} + 2q\frac{d^2T}{dy\,dz} + q^2\frac{d^2T}{dz^2} = 0,$$

$$\frac{dT}{dz} - \frac{d}{dq}\left(\frac{dT}{dy}\right) + T\frac{d}{dp}\left(\frac{dT}{dy}\right) - \left(\frac{dT}{dy}\right)\frac{dT}{dp} = 0,$$

while (4) becomes

$$dp + Tdq - \left(\frac{dT}{dy}\right)(dz - qdy) = \mu df.$$

Case IV. $T=0$. We have in this case to deal with a transformation of the very important equation

$$F\{s, p, q, z, x, y\}=0$$

discussed by Ampère.

Here (6) becomes $\qquad \Sigma=\dfrac{dV}{dq}$,

and (8) $\qquad \dfrac{d\Sigma}{dq}, =\dfrac{d^2V}{dq^2}, =0.$

Hence V is linear in q, and (1) may be written

$$s+Qq+Z=0 \text{ .... } (9),$$

Q, Z, being functions of x, y, z, p. Taking this form, we have

$$\Sigma=Q,$$
$$V-q\Sigma=Z.$$

The equations (5) now reduced to one only, viz.

$$\frac{dQ}{dy}-\frac{dZ}{dz}-Z\frac{dQ}{dp}+Q\frac{dZ}{dp}=0,$$

and the first integral is found from the equation

$$dp+Qdz+Zdy=\mu df.$$

These results agree with those given by Ampère.

September, 1877.

SUGGESTION OF A MECHANICAL INTEGRATOR FOR THE CALCULATION OF $\int(Xdx+Ydy)$ ALONG AN ARBITRARY PATH.*

By Professor *Cayley*.

I CONSIDER an integral $\int(Xdx+Ydy)$, where X, Y are each of them a given function of the variables (x, y); $Xdx+Ydy$ is thus not in general an exact differential; but

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* Read at the British Association Meeting at Plymouth, August 20, 1877.

assuming a relation between (x, y), that is a path of the integral, there is in effect one variable only, and the integral becomes calculable. I wish to show how for any given values of the functions X, Y, but for an arbitrary path, it is possible to construct a mechanism for the calculation of the integral: viz. a mechanism, such that a point D thereof being moved in a plane along a path chosen at pleasure, the corresponding value of the integral shall be exhibited on a dial.

The mechanism (for convenience I speak of it as actually existing) consists of a square block or inverted box, the upper horizontal face whereof is taken as the plane of xy, the equations of its edges being $y=0$, $y=1$, $x=0$, $x=1$ respectively. In the wall faces represented by these equations, we have the endless bands A, A', B, B' respectively; and in the plane of xy, a driving point D, the coordinates of which are (x, y), and a regulating point R, mechanically connected with D, in suchwise that the coordinates of R are always the given functions X, Y of the coordinates of D;* the nature of the mechanical connexion will of course depend upon the particular functions X, Y.

This being so, D drives the bands A and B in such manner that to the given motions dx, dy of D corresponds a motion dx of the band A, and a motion dy of the band B; A drives A' with a velocity-ratio depending on the position of the regulator R in suchwise that the coordinates of R being X, Y, then to the motion dx of A corresponds a motion Xdx of A'; and, similarly, B drives B' with a velocity-ratio depending on the position of R, in suchwise that to the motion dy of B corresponds a motion Ydy of B'. Hence, to the motions dx, dy of the driver D, there correspond the motions Xdx and Ydy of the bands A' and B' respectively; the band A' drives a hand or index, and the band B' drives in the contrary sense a graduated dial, the hand and dial rotating independently of each other about a common centre; the increased reading of the hand on the dial is thus $=Xdx+Ydy$; and supposing the original reading to be zero, and the driver D to be moved from its original position along an arbitrary path to any other position whatever, the reading on the dial will be the corresponding value of the integral $\int(Xdx+Ydy)$.

It is obvious that we might, by means of a combination

* It might be convenient to have as the coordinates of R, not X, Y but ξ, η, determinate functions of X, Y respectively.

of two such mechanisms, calculate the value of an integral $\int f(u)\,du$ along an arbitrary path of the complex variable u, $=x+iy$; in fact, writing $f(x+iy)=P+iQ$, the differential is $(P+iQ)(dx+idy)$, $=Pdx-Qdy+i(Qdx+Pdy)$; and we thus require the calculation of the two integrals $\int(Pdx-Qdy)$ and $\int(Qdx+Pdy)$, each of which is an integral of the above form. Taking for the path a closed curve, it would be very curious to see the machine giving a value zero or a value different from zero, according as the path included or did not include within it a critical point; it seems to me that this discontinuity would really exhibit itself without the necessity of any change in the setting of the machine.

The ordinary modes of establishing a continuously-variable velocity-ratio between two parts of a machine depend upon friction; and in particular this is the case in Prof. James Thomson's mechanical integrator—there is thus of course a limitation of the driving power. It seems to me that a variable velocity-ratio, the variation of which is practically although not strictly continuous, might be established by means of toothed wheels (and so with unlimited driving power) in the following manner.

Consider a revolving wheel A, which by means of a link BC, pivoted to a point B of the wheel A and a point C of a toothed wheel or arc D, communicates a reciprocating motion to D; the extent of this reciprocating motion depending on the distance of B from the centre of A, which distance, or say the half-throw, is assumed to be variable. Here during a half-revolution of A, D moves in one direction, say upwards; and during the other half revolution of A, D moves in the other direction, say downwards; the extent of these equal and opposite motions varying with the throw. Suppose then that D works a pinion E, the centre of which is not absolutely fixed but is so connected with A that during the first half revolution of A (or while D is moving upwards), E is in gear with D, and during the second half revolution of A, or while D is moving downwards, E is out of gear with D; the continuous rotation of A will communicate an intermittent rotation to E, in such manner nevertheless, that to each entire revolution of A or rotation through the angle 2π there will (the throw remaining constant) correspond a rotation of E through the angle $n.2\pi$, where the coefficient n depends upon the throw.* And evidently if A be driven by

* If instead of the wheel or arc D with a reciprocating circular motion, we have a double rack D with a reciprocating rectilinear motion, such that the wheel E is

a wheel A', the angular velocity of which is $\frac{1}{\lambda}$ times that of A, then to a rotation of A' through each angle $\frac{2\pi}{\lambda}$, there will correspond an entire revolution of A, and therefore, as before, a rotation of E through the determinate angle $n.2\pi$; hence, λ being sufficiently large to each increment of rotation of A', there corresponds in E an increment of rotation which is $n\lambda$ times the first-mentioned increment; viz. E moves (intermittently and possibly also with some "loss of time" on E coming successively in gear and out of gear with D, or in beats as explained) with an angular velocity which is $=n\lambda$ times the angular velocity of A'. And thus the throw, and therefore n being variable, the velocity-ratio $n\lambda$ is also variable.

We may imagine the wheel A as carrying upon it a piece L sliding between guides, which piece L carries the pivot B, of the link BC, and works by a rack on a toothed wheel α concentric with A, but capable of rotating independently thereof. Then if α rotates along with A, as if forming one piece therewith, it will act as a clamp upon L, keeping the distance of B from the centre of A, that is the half-throw, constant; whereas, if α has given to it an angular velocity different from that of A, the effect will be to vary the distance in question; that is to vary the half-throw, and consequently the velocity-ratio of A and E. And, in some such manner, substituting for A and E the bands A and A' of the foregoing description, it might be possible to establish between these bands the required variable velocity-ratio.

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placed between the two racks, and is in gear on the one side with one of them when the rack is moving upwards, and on the other side with the other of them when the rack is moving downwards; then the continuous circular motion of A will communicate to E a continuous circular motion, not of course uniform, but such that to each entire revolution of A or rotation through the angle 2π, there will correspond a rotation of E through an angle $n.2\pi$ as before. This is in fact a mechanical arrangement made use of in a mangle, the double rack being there the follower instead of the driver.

TRANSACTIONS OF SOCIETIES.

The Meeting of the British Association at Plymouth.

The forty-seventh meeting of the British Association for the Advancement of Science was opened at Plymouth on Wednesday, August 15, 1877, under the Presidency of Professor Allen Thomson. The Sections assembled on Thursday morning, August 16, the following being the list of officers of Section A (Mathematical and Physical Science):

President.—Professor G. C. Foster, F.R.S.

Vice-Presidents.—Professor J. C. Adams, F.R.S.; Professor W. G. Adams, F.R.S.; Professor Cayley, F.R.S.; Professor S. Haughton, F.R.S.; Professor Bartholomew Price, F.R.S.; Lord Rayleigh, F.R.S.; Sir W. Thomson, F.R.S.

Secretaries.—Professor Barrett; J. T. Bottomley; J. W. L. Glaisher, F.R.S.; F. G. Landon.

Among the mathematicians present at the meeting, besides those mentioned among the officers, were Professor R. S. Ball, F.R.S.; Professor D. Bierens de Haan: Professor J. D. Everett; Mr. R. B. Hayward, F.R.S.; Mr. W. M. Hicks; Mr. H. M. Jeffery; Mr. T. P. Kirkman, F.R.S.; Professor C. J. Lambert; Mr. C. W. Merrifield, F.R.S.; and Professor R. K. Miller.

In his inaugural address Professor Carey Foster chiefly treated of the relations between Mathematics and Physics, considered from a physical point of view. The Section did uot meet on the Saturday, but on Monday, August 20, a department of mathematics sat under the Presidency of Professor Cayley, when the following papers were read:

J. W. L. Glaisher.—Report of the Committee on Mathematical Tables.

Prof. D. B. de Haan.—On the Variation of the Modulus in Elliptic Integrals.

Prof. J. C. Adams.—On the Calculation of Bernoulli's Numbers up to B_{62} by means of Staudt's Theorem.

Prof. J. C. Adams.—On the Calculation of the Sum of the Reciprocals of the First Thousand Integers, and on the Value of Euler's Constant to 260 Places of Decimals.

Prof. J. C. Adams.—On a Simple Proof of Lambert's Theorem.

H. M. Jeffery.—On Cubics of the Third Class with Three Single Foci.

H. M. Jeffery.—On a Cubic Curve Referred to a Tetrad of Corresponding Points or Lines.

Sir William Thomson.—Solutions of Laplace's Tidal Equation for certain Special Types of Oscillation.

Prof. Cayley.—Suggestion of a Mechanical Integrator for the Calculation of an integral $\int(Xdx + Ydy)$ along an Arbitrary Path.

J. W. L. Glaisher.—On the Values of a Class of Determinants.

J. W. L. Glaisher.—On the Enumeration of the Primes in Burckhardt's and Dase's Tables.

F. G. Landon.—On a method of deducing the sum of the reciprocals of the first $2^{p}n$ numbers from the sum of the reciprocals of the first n numbers.

Besides these, the following papers relating to mathematical subjects were also read during the meeting:—

Prof. J. C. Adams.—On some recent advances in the Lunar Theory.

Prof. S. Haughton.—The Solar Eclipse of Agathocles, considered in reply to Professor Newcomb's criticism on the coefficient of the acceleration of the moon's mean motion.

Prof. S. Haughton.—Summary of the first reduction of the Tidal Observations made by the recent Arctic Expedition.

Prof. Osborne Reynolds.—On the rate of progression of groups of waves and the rate at which energy is transmitted by waves.

The sum of £100 was granted to the Committee on Mathematical Tables for commencing the calculation of a factor table for the fourth million, in continuation of Burckhardt's tables of the first three millions. Dase's tables extend from 6,000,000 to 9,000,000 so that there is a gap of three millions which it is now proposed to fill up. The whole sum granted on the recommendation of Section A was £240. A committee was appointed to invite reports on different special branches of science.

The Association adjourned on Wednesday, August 22, till Wednesday, August 14, 1878, at Dublin. Mr. W. Spottiswoode is the President elect. In 1879 the meeting of the Association will be held at Nottingham; and it is probable that the meeting will be held in 1880 at Swansea, and in 1881 at York, the city where the first meeting of the Association took place fifty years before. J. W. L. G.

ON A SIMPLE PROOF OF LAMBERT'S THEOREM.

By Professor *J. C. Adams, M.A., F.R.S.*

THE following proof of Lambert's theorem, which I find among my old papers, appears to be as simple and direct as can be desired.

Let a denote the semi-axis major and e the eccentricity of an elliptic orbit, n the mean motion and μ the absolute force.

Also let r, r' denote the radii vectores, and u, u' the eccentric anomalies at the extremities of any arc, k the chord, and t the time of describing the arc.

Then $r = a(1 - e\cos u)$, $r' = a(1 - e\cos u')$,

$$k^2 = a^2(\cos u - \cos u')^2 + a^2(1 - e^2)(\sin u - \sin u')^2,$$

and $$nt = \left(\frac{\mu}{a^3}\right)^{\frac{1}{2}} t = u - u' - e(\sin u - \sin u');$$

or $$\frac{r + r'}{2a} = 1 - \left(e\cos\frac{u + u'}{2}\right)\cos\frac{u - u'}{2},$$

$$\frac{k^2}{4a^2} = \sin^2\frac{u + u'}{2}\sin^2\frac{u - u'}{2} + (1 - e^2)\cos^2\frac{u + u'}{2}\sin^2\frac{u - u'}{2}$$

$$= \sin^2\frac{u - u'}{2}\left(1 - e^2\cos^2\frac{u + u'}{2}\right);$$

and $$nt = u - u' - 2\left(e\cos\frac{u + u'}{2}\right)\sin\frac{u - u'}{2}.$$

Hence we see that if a, and therefore also n, be given then $r + r'$, k and t are functions of the two quantities $u - u'$ and $e\cos\dfrac{u + u'}{2}$.

Let $u - u' = 2\alpha$ and $e\cos\dfrac{u + u'}{2} = \cos\beta$, then

$$\frac{r + r'}{2a} = 1 - \cos\alpha\cos\beta,$$

$$\frac{k}{2a} = \sin\alpha\sin\beta;$$

therefore $$\frac{r + r' + k}{2a} = 1 - \cos(\beta + \alpha),$$

and $$\frac{r+r'-k}{2a} = 1 - \cos(\beta - \alpha);$$

also $$nt = 2\alpha - 2\sin\alpha\cos\beta$$
$$= \{\beta + \alpha - \sin(\beta + \alpha)\} - \{\beta - \alpha - \sin(\beta - \alpha)\}.$$

The first two of these equations give $\beta + \alpha$ and $\beta - \alpha$ in terms of $r + r' + k$ and $r + r' - k$, and the third is the expression of Lambert's theorem.

An exactly similar proof may be given in the case of an hyperbolic orbit.

Let $\frac{1}{2}(e^u + e^{-u})$ be denoted by $\mathrm{csh}(u)$,

and $\frac{1}{2}(e^u - e^{-u})$ by $\mathrm{snh}(u)$,

which quantities may be called the hyperbolic cosine and hyperbolic sine of u. Then we have

$$\mathrm{csh}^2(u) - \mathrm{snh}^2(u) = 1,$$

$$\mathrm{csh}(u) + \mathrm{csh}(u') = 2\,\mathrm{csh}\frac{u+u'}{2}\,\mathrm{csh}\frac{u-u'}{2},$$

$$\mathrm{csh}(u) - \mathrm{csh}(u') = 2\,\mathrm{snh}\frac{u+u'}{2}\,\mathrm{snh}\frac{u-u'}{2},$$

$$\mathrm{snh}(u) - \mathrm{snh}(u') = 2\,\mathrm{cnh}\frac{u+u'}{2}\,\mathrm{snh}\frac{u-u'}{2}.$$

The coordinates of any point in the hyperbola referred to its axes may be represented by

$$x = a\,\mathrm{cnh}(u),$$
$$y = a\sqrt{(e^2 - 1)}\,\mathrm{snh}(u).$$

If u, u' denote the values of u corresponding to the two extremities of the arc, we have

$$r = a\{e\,\mathrm{csh}(u) - 1\}, \quad r' = a\{e\,\mathrm{csh}(u') - 1\},$$

$$k^2 = a^2\{\mathrm{csh}(u) - \mathrm{csh}(u')\}^2 + a^2(e^2 - 1)\{\mathrm{snh}(u) - \mathrm{snh}(u')\}^2;$$

or $$\frac{r+r'}{2a} = \left(e\,\mathrm{csh}\frac{u+u'}{2}\right)\mathrm{csh}\frac{u-u'}{2} - 1,$$

$$\frac{k^2}{4a^2} = \mathrm{snh}^2\frac{u-u'}{2}\left(e^2\,\mathrm{csh}^2\frac{u+u'}{2} - 1\right).$$

Also, twice the area of the sector limited by r and r'

$$=a^2 \sqrt{(e^2-1)}\,\{e(\operatorname{snh} u - \operatorname{snh} u') - (u-u')\}$$

$$=a^2 \sqrt{(e^2-1)}\left\{2\left(e \operatorname{csh}\frac{u+u'}{2}\right)\operatorname{snh}\frac{u-u'}{2} - (u-u')\right\};$$

and twice the area described in a unit of time is

$$\sqrt{\{\mu a(e^2-1)\}}.$$

Hence $t=\left(\dfrac{a^3}{\mu}\right)^{\frac{1}{2}}\left\{2\left(e\operatorname{csh}\dfrac{u+u'}{2}\right)\operatorname{snh}\dfrac{u-u'}{2}-(u-u')\right\}$,

and therefore if a be given, $r+r'$, k and t are functions of the two quantities $e \operatorname{csh}\dfrac{u+u'}{2}$ and $u-u'$.

Let $u-u'=2\alpha$ and $e\operatorname{csh}\dfrac{u+u'}{2}=\operatorname{csh}(\beta)$, which is always possible since e is greater than 1, then

$$\frac{r+r'}{2a}=\operatorname{csh}(\beta)\operatorname{csh}(\alpha)-1,$$

$$\frac{k}{2a}=\operatorname{snh}(\beta)\operatorname{snh}(\alpha);$$

therefore $\dfrac{r+r'+k}{2a}=\operatorname{csh}(\beta+\alpha)-1,$

and $\dfrac{r+r'-k}{2a}=\operatorname{csh}(\beta-\alpha)-1.$

Also $t=\left(\dfrac{a^3}{\mu}\right)^{\frac{1}{2}}\{2\operatorname{csh}(\beta)\operatorname{snh}(\alpha)-2\alpha\}$

$$=\left(\frac{a^3}{\mu}\right)^{\frac{1}{2}}\{\operatorname{snh}(\beta+\alpha)-(\beta+\alpha)-\operatorname{snh}(\beta-\alpha)+(\beta-\alpha)\}.$$

As before, the first two of these equations give $\beta+\alpha$ and $\beta-\alpha$ in terms of $r+r'+k$ and $r+r'-k$, and the last is the expression of Lambert's theorem in the case of the hyperbola.

When the orbit is parabolic a becomes infinite, and since $r+r'$ and k are finite, the quantities α and β become indefinitely small.

Hence $\dfrac{r+r'+k}{2a}=1-\cos(\beta+\alpha)=\frac{1}{2}(\beta+\alpha)^2$ ultimately,

$$\frac{r+r'-k}{2a}=1-\cos(\beta-\alpha)=\tfrac{1}{2}(\beta-\alpha)^2 \text{ ultimately.}$$

Also $$t=\left(\frac{a^3}{\mu}\right)^{\frac{1}{2}}\{\beta+\alpha-\sin(\beta+\alpha)-(\beta-\alpha)+\sin(\beta-\alpha)\}$$

$$=\left(\frac{a^3}{\mu}\right)^{\frac{1}{2}}\{\tfrac{1}{6}(\beta+\alpha)^3-\tfrac{1}{6}(\beta-\alpha)^3\} \text{ ultimately}$$

$$=\tfrac{1}{6}\left(\frac{a^3}{\mu}\right)^{\frac{1}{2}}\left\{\left(\frac{r+r'+k}{a}\right)^{\frac{3}{2}}-\left(\frac{r+r'-k}{a}\right)^{\frac{3}{2}}\right\} \text{ ultimately}$$

$$=\frac{1}{6\sqrt{\mu}}\{(r+r'+k)^{\frac{3}{2}}-(r+r'-k)^{\frac{3}{2}}\},$$

which is Lambert's theorem in the case of the parabola.

ON THE WAVE SURFACE.

By *A. Mannheim*, Professor at the Ecole Polytechnique of Paris.

THE wave surface cuts each of its principal planes and the plane at infinity in a conic and a circle. In each of these planes the circle and the conic have four points of intersection, which are the conical points of the wave surface. We shall show that it is easy to deduce from these well-known results that there are certain planes touching the wave surface along lines which are circles, and that *planes parallel to these cut it in curves which are anallagmatics of the fourth order.*

Fig. (6) shows the traces of the wave surface in the principal planes, the circles being represented by dotted lines, and the extremities of the axes being denoted by letters which also denote their lengths. In the plane of yz the conical points are imaginary, but they lie upon the common real chords of the circle and the conic in this plane.

One of these chords el is parallel to the axis of y and is at a distance from this axis equal to $b\left(\frac{a^2-c^2}{a^2-b^2}\right)^{\frac{1}{2}}$. Also in the plane of xy we have the chord dm parallel to the axis of y and at a distance from this axis equal to $b\left(\frac{a^2-c^2}{b^2-c^2}\right)^{\frac{1}{2}}$.

Every plane passing through one or other of these straight lines cuts the wave surface in a curve having two double points. The wave surface is therefore cut by the plane containing the two straight lines el, dm in a curve of the fourth order which has four double points, viz., in a curve which is decomposable into two conics.

But from the values of oe and od it follows that

$$\frac{1}{oe^2}+\frac{1}{od^2}=\frac{1}{b^2},$$

and consequently that *ed* is a tangent to the circle of radius *b*. The plane (*el*, *dm*) is therefore also tangent at *g* to the wave surface, and *g* is thus a double point of the section made by the plane. But this section consists of two conics: the point *g* must therefore be common to these conics. They thus have five common points and must be coincident. The plane (*el*, *dm*) therefore touches the wave surface along one of these conics. We shall show that these conics are circles.

Cut the wave surface by a plane perpendicular to the plane of *xz*. This plane cuts the plane of *xz* in a straight line which is an axis of the section determined by it in the wave surface, and it cuts the plane (*el*, *dm*) in a perpendicular to this axis. If we move this cutting plane parallel to itself to infinity, the curve of intersection is composed of a circle and a concentric conic, and the intersection of the plane (*el*, *dm*) with the plane at infinity, which is, as we have said, perpendicular to an axis of these curves, cannot be one of their common tangents but must be one of their common chords. Thus the singular tangent planes of the wave surface cut the plane at infinity along the common chords of the circle and the conic situated in that plane. Consequently the conics along which they touch the wave surface pass through points situated on the imaginary circle at infinity, and are therefore circles. As the wave surface has a centre, its singular tangent planes are symmetrical two and two. There are then two singular tangent planes for each of the chords common to the circle and the conic, situated in the plane at infinity, viz., the wave surface has twelve singular tangent planes. Of these twelve tangent planes, only four are real, since on the plane at infinity there are only two real common chords.

Planes parallel to the singular tangent planes pass through the chords common to the circle and the conic situated on the plane at infinity, they therefore cut the wave surface in curves, having for double points the conical points situated on these chords. And as these double points are upon the circle at infinity we see that *planes parallel to the singular tangent planes cut the wave surface along anallagmatics of the fourth order**

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* I have already proved, in a very different manner, a particular case of this theorem in a communication made to the French Association for the Advancement of Science at the meeting at Nantes (1875).

ON LONG SUCCESSIONS OF COMPOSITE NUMBERS.

By *J. W. L. Glaisher.*

§ 1. Burckhardt's tables (1814–1817) give the least factor of every number up to 3,036,000, and Dase's tables (1862–1865), the least factor of every number from 6,000,000 to 9,000,000; the primes in these tables are marked by a dash, or short line, facing them. The number of primes between any limits can therefore be obtained by counting the number of dashes; and in this manner I effected the enumeration of the primes in the six millions over which the tables extend.*

Calling, for convenience of expression, the hundred numbers between $100n$ and $100(n+1)$ the $(n+1)^{\text{th}}$ *century*, then the enumeration was made by centuries, that is to say, the number of primes in each century was obtained by counting and entered in its proper place upon a printed form.

The following specimen exhibits five columns of one of the tables so formed:

No. 270.

20	9	30	6	40	6	50	10	60	9
21	7	31	7	41	9	51	6	61	8
22	6	32	3	42	6	52	7	62	5
23	6	33	5	43	6	53	6	63	6
24	4	34	9	44	5	54	5	64	6
25	2	35	5	45	8	55	7	65	6
26	7	36	10	46	4	56	8	66	8
27	9	37	7	47	9	57	3	67	7
28	8	38	5	48	7	58	8	68	3
29	4	39	7	49	17	59	9	69	8
	62		64		77		69		66

This portion of the table shows that the number of the primes between 2,702,000 and 2,702,100 is 9, between 2,702,100 and 2,702,200 is 7, and so on; or, in other words, that the $27{,}021^{\text{th}}$ century contains 9 primes, the $27{,}022^{\text{th}}$ contains 7 primes, and so on. The results were then classified in tables;

* A preliminary account of the results is printed in the *Proceedings of the Cambridge Philosophical Society*, vol. III., pp. 17–23, 47–56, (1876–1877).

and those published in the *Proceedings of the Cambridge Philosophical Society* give the number of centuries, in each group of 100,000 in the six millions, which contain n primes for n as argument. Thus, for example, the eighth column of the table for the third million shows that in the thousand centuries between 2,700,000 and 2,800,000 there is no century that contains no prime, 2 centuries each of which contains one prime, 7 centuries each of which contains two primes, and so on, there being 195 centuries containing six primes and one century containing as many as seventeen primes.

The following is a similar table, in which each column has reference to a million:

Number of centuries, each of which contains n primes.

n	first million.	second million.	third million.	seventh million.	eighth million.	ninth million.
0	0	1	1	6	4	4
1	3	16	25	28	30	34
2	29	72	97	173	171	178
3	140	257	338	482	541	570
4	372	667	775	1,049	1,066	1,078
5	801	1,253	1,408	1,603	1,691	1,742
6	1,362	1,743	1,878	1,948	1,993	1,966
7	1,765	2,032	1,997	1,916	1,754	1,788
8	1,821	1,612	1,526	1,366	1,394	1,278
9	1,554	1,182	1,036	840	787	778
10	1,058	691	558	374	360	390
11	592	311	227	156	155	143
12	316	113	98	46	40	38
13	122	39	28	10	10	11
14	32	7	6	3	2	2
15	20	3	1	0	2	0
16	8	1	0	0	0	0
17	3	0	1†	0	0	0
21	1	0	0	0	0	0
26	1*	0	0	0	0	0
No. of primes.	78,499	70,433	67,885	63,799	63,158	62,760

§2. Whenever there is no prime in a century there must be a succession of at least 100 consecutive composite numbers, and whenever there is only one prime there must be a succession of at least 50, but, of course, a 2-prime century or a

* 1 and 2 are both counted as primes.

† The specimen on the preceding page (which is half of one of the small tables, and therefore $\frac{1}{1200}$ of the whole table) was chosen so as to include this (the 270,50th) century. It will be noticed that there is no other century in the second or third million that contains so many as 17 primes, and that in the third million there is no century containing 16 primes, and only one in the second million.

3-prime century need not indicate a long sequence. In order to find long successions of composite numbers, I had all the instances in which there were 0, 1, 2 or 3 primes in a century looked out in the tables, and the cases noted down in which the sequence was 50 or upwards. The number of sequences exceeding 50 was very great,* and therefore in the following list I only give the sequences of 79 and upwards in the first million, and of 99 and upwards in the second and third millions, that were thus found:

FIRST MILLION.—*Sequences of* 79 *and upwards.*

Lower Limit.	Upper Limit.	Sequence.
155,921	156,007	85
338,033	338,119	85
370,261	370,373	111
461,717	461,801	83
492,113	492,227	113
544,279	544,367	87
604,073	604,171	97
682,819	682,901	81
815,729	815,809	79
818,723	818,813	89
822,433	822,517	83
838,249	838,349	99
843,911	844,001	89
863,393	863,479	85
927,869	927,961	91
979,567	979,651	83
982,981	983,063	81

SECOND MILLION.—*Sequences of* 99 *and upwards.*

Lower Limit.	Upper Limit.	Sequence.
1,349,533	1,349,651	117
1,357,201	1,357,333	131
1,388,483	1,388,587	103
1,444,309	1,444,411	101
1,468,277	1,468,387	109
1,561,919	1,562,051	131
1,648,081	1,648,181	99
1,655,707	1,655,807	99
1,664,123	1,664,227	103
1,671,781	1,671,907	125
1,761,187	1,761,289	101
1,775,069	1,775,171	101
1,895,359	1,895,479	119

* Tables containing these sequences were exhibited to the Meeting of the London Mathematical Society, May 10, 1877 (see *Messenger, ante* p. 56).

THIRD MILLION.—*Sequences of* 99 *and upwards.*

Lower Limit.	Upper Limit.	Sequence.
2,010,733	2,010,881	147
2,238,823	2,238,931	107
2,243,987	2,244,091	103
2,305,169	2,305,271	101
2,314,439	2,314,547	107
2,345,989	2,346,089	99
2,597,981	2,598,091	109
2,614,883	2,614,987	103
2,637,799	2,637,911	111
2,784,373	2,784,473	99
2,867,107	2,867,213	105
2,898,239	2,898,359	119

Thus the 85 numbers between the primes 155,921 and 156,007 are composite, and so on, viz. the numbers in the first two columns are primes, and the numbers intermediate to the lower limit and the upper limit are all composite.

It is to be specially noted that the above list is not complete, as, of course, long sequences may occur when the number of primes in adjacent centuries is 4 or more. But the parts of the table where there were but few primes in a century were naturally most likely to be fruitful in long sequences of composite numbers. I ought also to state that I am not *certain* even that all the sequences obtained from the 0's, 1's, 2's, 3's are included, as the work was merely performed once, and in view of the necessary incompleteness of the list, it did not seem worth while to undertake a thorough examination or a duplicate calculation, in order to prove that no sequence had been omitted. The list is, therefore, merely a collection of long successions of composite numbers, without regard to completeness. I have, of course, verified that all the sequences given are correct.

It may be noted that the longest sequence (147) was obtained from a 3-prime century, the two sequences of 131 in the second million were each obtained from 1-prime centuries, and in the first million the sequence of 113 resulted from a 2-prime century, and that of 111 from a 3-prime century. The 0-prime centuries produced sequences of 125 and 111.

Although a 3-prime century is less likely to produce a long sequence than a 2-prime century, still the number of the former is so much greater that the sequences obtained from them are almost equal in importance to those derived

from the 2-prime centuries; and a similar remark will apply, though probably in a less degree, to the 4-prime and other centuries, which were not examined.

§3. The great length of some of the sequences and the number of long sequences are, I think, remarkable; and the most noticeable of all is the sequence of 111, which occurs so early as 370,261.

It is known that any sequence of composite numbers, however long, *must* occur at a certain definite place in the series of natural numbers, for if p be any prime, q the next prime to p, and 2, 3, $5 \ldots p$ all the primes up to p, then the $q-2$ numbers immediately following $(2.3.5.7 \ldots p)+1$ must be composite, for $(2.3.5 \ldots p)+2$ is divisible by 2, the next number by 3, the next by 2, and so on, every number obviously having a divisor, up to and including $(2.3.5 \ldots p)+(q-1)$. The $q-2$ numbers between $(2.3.5 \ldots p)+1$ and $(2.3.5 \ldots p)+q$ must therefore be composite, and the limiting numbers may be either prime or composite, so that we have a sequence of at least $q-2$ composite numbers.

The prime next below 111 is 109, and the next prime is 113, so that of necessity the 111 numbers following

$$(2.3.5 \ldots 109)+1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{i})$$

are composite; we therefore have a sequence of 111 immediately following

$$279,734,996,817,854,936,178,276,161,872,067,809,674,997,231 \ldots\ldots\ldots\ldots\ldots(\text{ii}),$$

this number being the value of (i).

The number (ii) consists of 45 figures, and is enormously greater than 370,261, where a sequence of 111 actually occurs, so that long sequences are met with far earlier than the theorem requires them to happen. In fact,

$$2.3.5 \ldots 17 = 510,510$$

$$2.3.5 \ldots 19 = 9,699,690,$$

so that we cannot *predict* from the theorem a sequence of more than 21 in the first ten millions. The sequences for the seventh, eighth, and ninth millions, as also some remarks upon the sequences that occur between 1 and 30,000, will be given in a continuation of this paper.

NOTE ON A DIFFERENTIAL EQUATION.

By *H. W. Lloyd Tanner, M.A.*

THE equation is

$$\frac{dz_1}{dx_1} - \frac{dz_2}{dx_2} + \ldots (-)^{n-1} \frac{dz_n}{dx_n} = 0 \ldots\ldots\ldots\ldots\ldots(1).$$

The following form of solution may be worth record.

Let Δ_m represent the determinant formed by omitting the m^{th} column of the system

$$\begin{array}{l} \frac{d\phi_1}{dx_1}, \frac{d\phi_1}{dx_2}, \ldots \frac{d\phi_1}{dx_n}, \\ \frac{d\phi_2}{dx_1}, \ldots\ldots\ldots \frac{d\phi_2}{dx_n}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \frac{d\phi_{n-1}}{dx_1}, \ldots\ldots\ldots \frac{d\phi_{n-1}}{dx_n}, \end{array}$$

where $\phi_1 \ldots \phi_{n-1}$ are arbitrary functions of $x_1 \ldots x_n$. Then the general solution of (1) is

$$z_m = \Delta_m \; (m = 1, 2, \ldots n) \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

That (2) is a solution of (1) follows if we can prove

$$\frac{d\Delta_1}{dx_1} - \frac{d\Delta_2}{dx_2} + \ldots (-)^{n-1} \frac{d\Delta_n}{dx_n} = 0 \ldots\ldots\ldots\ldots(3),$$

or, as it may be written,

$$\left| \begin{array}{l} \frac{d}{dx_1}, \frac{d}{dx_2}, \ldots \frac{d}{dx_n} \\ \frac{d\phi_1}{dx_1}, \frac{d\phi_1}{dx_2}, \ldots \frac{d\phi_1}{dx_n} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \frac{d\phi_{n-1}}{dx_1}, \frac{d\phi_{n-1}}{dx_2}, \ldots \frac{d\phi_{n-1}}{dx_n} \end{array} \right| = 0 \ldots\ldots\ldots\ldots(4).$$

Now if (3) or (4) be expanded there will be in each term one, and only one, factor of the form $\dfrac{d^2\phi_k}{dx_l dx_m}$. This factor will occur in two terms, multiplied by the same coefficient, but

with opposite signs. For instance, take $\dfrac{d^2\phi_1}{dx_1 dx_2}$. The coefficient of this in each of the two terms in which it occurs is the Jacobian $\dfrac{d(\phi_2,\ldots\phi_{n-1})}{d(x_3,\ldots x_n)}$; but one term is positive and the other negative. Hence, as there are no terms in the expansion of (4) without a factor of the second order, and terms containing such factors annul each other, (3) and (4) are identically true; or (2) is a solution of (1).

It is also the general solution of (1). To prove this we show that the equations

$$z_m = \lambda\Delta_m \; (m = 1, 2, \ldots n) \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

impose no restriction upon z_m, and then making these substitutions in (1) we determine the general form of λ. It will be found that λ is such that (5) is equivalent to (2).

First then, (5) do not imply any relation between $z_1, \ldots z_n$, in other words $\phi_1, \ldots \phi_{n-1}$, λ can be determined so as to satisfy (5) whatever $z_1 \ldots z_n$ may be. In fact, the ϕ's are any $(n-1)$ independent particular integrals of the equation

$$z_1 \frac{d\phi}{dx_1} - z_2 \frac{d\phi}{dx_2} + \ldots (-)^{n-1} z_n \frac{d\phi}{dx_n} = 0 \ldots\ldots\ldots\ldots (6),$$

which can be found by integrating the system

$$\frac{dx_1}{z_1} = -\frac{dx_2}{z_2} = \ldots = (-)^{n-1} \frac{dx_n}{z_n}.$$

For, calling these particular integrals, $\phi \ldots \phi_{n-1}$, the general value of ϕ is

$$\phi = f(\phi_1, \phi_2 \ldots \phi_{n-1});$$

so that ϕ satisfies the equation

$$\frac{d(\phi, \phi_1, \phi_2 \ldots \phi_{n-1})}{d(x_1, x_2, \ldots x_n)} = 0,$$

that is
$$\Delta_1 \frac{d\phi}{dx_1} - \Delta_2 \frac{d\phi}{dx_2} + \ldots = 0.$$

Comparing this with (6) we reproduce (5).

Next suppose that $z_1 \ldots z_n$ satisfy (1). As has just been shown, the substitutions (5) can be made without loss of generality, so that (1) is equivalent to

$$\lambda \left\{ \frac{d\Delta_1}{dx_1} - \frac{d\Delta_2}{dx_2} + \&c. \right\} + \Delta_1 \frac{d\lambda}{dx_1} - \Delta_2 \frac{d\lambda}{dx_2} + \ldots = 0.$$

The first term of this equation vanishes by (3). The second becomes

$$\frac{d(\lambda, \phi_1, \dots \phi_{n-1})}{d(x_1, x_2 \dots x_n)} = 0,$$

or

$$\lambda = \psi'(\phi_1, \dots \phi_{n-1})$$

$$= \frac{d}{d\phi_1} \psi(\phi_1 \dots \phi_{n-1}) \text{ say.}$$

Substituting in (5) we get

$$z_m = \lambda \Delta_m = \frac{d\psi}{d\phi_1} \Delta_m$$

$$= \frac{d(\psi, \phi_2, \dots \phi_{n-2})}{d(x_1, x_2 \dots x_1)};$$

but this is not more general than Δ_m, because ϕ_1 is arbitrary as well as ψ. Hence (5) is equivalent to (2) when (1) is satisfied, and (2) is accordingly the general solution of (1).

Of course many equations of a more complicated form can be reduced to (1) by a suitable change of variables. For example, if A_1, B_1... be functions of $z_1 \dots z_n$ only, the equation

$$A_1 \frac{dz_1}{dx_1} + A_2 \frac{dz_2}{dx_1} + \dots + A_n \frac{dz_n}{dx_1}$$

$$+ B_1 \frac{dz_1}{dx_2} + \dots\dots\dots\dots + B_n \frac{dz_n}{dx_2}$$

$$+ \dots\dots\dots\dots\dots\dots\dots\dots = 0 \quad \dots\dots\dots(7)$$

is reducible to (1) by a change of dependent variables only, provided

$$A_1 dz_1 + A_2 dz_2 + \dots + A_n dz_n = \lambda d\zeta_1$$

$$B_1 dz_1 + B_2 dz_2 + \dots + B_n dz_n = -\lambda d\zeta_2$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

(n equations).

This implies certain relations between the A's, B's, &c.; when these relations are satisfied (7) can be expressed in the form (1).

Again, certain non-linear equations are reducible to (1), of which the following is an example:

$$\frac{d(z_2, z_3)}{d(x_2, x_3)} - \frac{d(z_1, z_3)}{d(x_1, x_3)} + \frac{d(z_1, z_2)}{d(x_1, x_2)} = 0.$$

This may be written in the form

$$\frac{dx_1}{dz_1} - \frac{dx_2}{dz} + \frac{dx_3}{dz_3} = 0,$$

and its solution is

$$x_1 = \frac{d(\phi_1, \phi_2)}{d(z_2, z_3)} : x_2 = \frac{d(\phi_1, \phi_2)}{d(z_1, z_3)} : x_3 = \frac{d(\phi_1, \phi_2)}{d(z_1, z_2)},$$

ϕ_1, ϕ_2 being arbitrary functions of z_1, z_2, z_3.

To find the general form of equations reducible to (1) we may suppose

$$z_m = f_m\{\zeta_1, \zeta_2, \ldots \zeta_n, \xi_1, \xi_2, \ldots \xi_n\}$$

$$x_r = f_{n+r}\{\zeta_1, \zeta_2, \ldots \zeta_n, \xi_1, \ldots\ldots \xi_n\}$$

$$(m, r = 1, 2, \ldots n).$$

Making these substitutions in (1) and regarding $\zeta_1 \ldots \zeta_n$ as new dependent variables, $\xi_1 \ldots \xi_n$ as the new independent variables, we shall obtain the general form of equation sought.

It may be worth while to draw attention to the particular case in which $n = 2$. The identity (1) becomes

$$\frac{d^2\phi_1}{dx_1 dx_2} = \frac{d^2\phi_1}{dx_2 dx_1}.$$

In the same case we say that if

$$\frac{dz_1}{dx_1} = \frac{dz_2}{dx_2} \text{..........................} (8),$$

then $$z_1 = \frac{d\phi}{dx_2}, \quad z_2 = \frac{d\phi}{dx_1}.$$

The equation (8) is the condition that

$$z_1 dx_2 + z_2 dx_1$$

should be an exact differential, and that the expression

$$z_1 \frac{du}{dx_1} - z_2 \frac{du}{dx_2}$$

should be an exact Jacobian; viz. this is

$$\frac{d\phi}{dx_2} \cdot \frac{du}{dx_1} - \frac{d\phi}{dx_1} \cdot \frac{du}{dx_2}.$$

Similarly (1) is the condition that the expression

$$z_1 \frac{du}{dx_1} - z_2 \frac{du}{dx_2} + \ldots (-)^{n-1} z_n \frac{du}{dz_n}$$

should be an exact Jacobian, namely,

$$\frac{d(\phi_1, \ldots \phi_{n-1}, u)}{d(x_1, \ldots\ldots\ldots x_n)}.$$

October, 1877.

ON LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER.

By Professor *E. J. Nanson, M.A.*

CONSIDER first the general linear homogeneous partial differential equation of the first order in which the dependent variable does not explicitly occur. This equation is of the form

$$X_1 \frac{du}{dx_1} + \ldots + X_n \frac{du}{dx_n} = 0 \ldots\ldots\ldots\ldots\ldots (1),$$

when $X_1 \ldots X_n$ are any functions of the independent variables $x_1 \ldots x_n$. Let Δ denote the operation

$$X_1 \frac{d}{dx_1} + \ldots + X_n \frac{d}{dx_n},$$

so that the equation (1) can be written

$$\Delta u = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Introduce a new system of independent variables $y_1, \ldots y_n$ in place of $x_1, \ldots x_n$. Since

$$\frac{du}{dx} = \frac{du}{dy_1} \frac{dy_1}{dx} + \ldots + \frac{du}{dy_n} \frac{dy_n}{dx}$$

we have

$$\Delta u = (\Delta y_1) \frac{du}{dy_1} + \ldots + (\Delta y_n) \frac{du}{dy_n},$$

and the transformed equation is*

$$(\Delta y_1) \frac{du}{dy_1} + \ldots + (\Delta y_n) \frac{du}{dy_n} = 0 \ldots\ldots\ldots\ldots (3).$$

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* Cf. Boole, *Differential Equations*, Supplementary Volume, p. 69.

Now choose for $y_1, \dots y_{n-1}$ a set of $n-1$ independent ntegrals of the system of ordinary simultaneous differential equations

$$\frac{dx_1}{X_1} = \dots = \frac{dx_n}{X_n} \quad \dots\dots\dots\dots\dots\dots\dots\dots (4),$$

and let y_n be any function of $x_1, \dots x_n$, which is independent of $y_1, \dots y_{n-1}$. Then we have

$$\Delta y_1 = 0, \dots \Delta y_{n-1} = 0,$$

and as y_n is independent of $y_1 \dots y_{n-1}$ it cannot be an integral of (4) so that

$$\Delta y_n \text{ not} = 0 \dots\dots\dots\dots\dots\dots\dots\dots (5).$$

The equation (3) then takes the form

$$(\Delta y_n) \frac{du}{dy_n} = 0,$$

or in virtue of (5)

$$\frac{du}{dy_n} = 0 \dots\dots\dots\dots\dots\dots\dots\dots\dots (6).$$

Hence $$u = \phi(y_1, \dots y_{n-1}) \dots\dots\dots\dots\dots\dots\dots (7),$$

when ϕ is arbitrary. The process shows that this is the most general solution. Hence we have the well-known rule for integrating the equation (1).

"Construct the auxiliary equations (4) and integrate them in the form

$$y_1 = c_1, \dots y_{n-1} = c_{n-1},$$

then the solution required is

$$u = \phi(y_1, \dots y_{n-1})$$

when ϕ is arbitrary."

Consider next the ordinary form of the linear equation, viz.

$$P_1 p_1 + \dots + P p_n = P \dots\dots\dots\dots\dots\dots (8),$$

when $p = \frac{dz}{dx}$ and $P, P_1, \dots P_n$ are any functions of $z, x_1, x_1 \dots x_n$.

Let $$u \equiv f(z, x_1, \dots x_n) = 0$$

be any solution of the equation (8). Since

$$\frac{du}{dx} + \frac{du}{dz} p = 0,$$

the given equation takes the form

$$P_1 \frac{du}{dx_1} + \ldots + P_n \frac{du}{dx_n} + P \frac{du}{dz} = 0 \ldots\ldots\ldots\ldots (9),$$

which is an equation of the kind already considered.*

To solve this, we integrate the auxiliary equations

$$\frac{dx_1}{P_1} = \ldots = \frac{dx_n}{P_n} = \frac{dz}{P} \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

in the form $u_1 = c_1, \ldots u_n = c_n,$

then the solution of (9) is

$$u = \phi\,(u_1, u_2, \ldots u_n),$$

where ϕ is arbitrary, and the solution of (8) is

$$\phi\,(u_1, u_2, \ldots u_n) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (11).$$

The above proof is at once suggested by the passages in Boole, to which reference is made.

The theory of the elimination of an arbitrary function may also be presented in a simple manner by the process of changing the independent variables.

Thus, suppose it required to eliminate the arbitrary function from the equation

$$u = \phi\,(y_1, \ldots y_{n-1}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (7),$$

where $y_1, \ldots y_{n-1}$ are $n-1$ given functions of the independent variables $x_1, \ldots x_n$. Introduce in place of $x_1, \ldots x_n$ the new set of independent variables $y_1, \ldots y_n$, where y_n is any function which is independent of the given functions $y_1, \ldots y_{n-1}$. The equation (7) gives at once

$$\frac{du}{dy_n} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Now this is a linear homogeneous partial differential equation of the first order. If we change back to the original variables the equation will still have the same character. Thus we have a very simple proof of the proposition that the elimination of the arbitrary function from an equation of the form (7) always leads to a linear homogeneous partial differential equation of the first order.

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* Cf. Boole, l.c., p. 68.

Let us, however, actually construct this equation. By the ordinary processes of the Differential Calculus, we have

$$\frac{du}{dy_n} = \frac{du}{dx_1}\frac{dx_1}{dy_n} + \ldots + \frac{du}{dx_n}\frac{dx_n}{dy_n} \;\ldots\ldots\ldots\ldots(12),$$

where $\dfrac{dx_1}{dy_n}, \ldots \dfrac{dx_n}{dy_n}$ have to be determined by n^2 equations of the form

$$\frac{dx_r}{dy_1}\frac{dy_1}{dx_s} + \ldots + \frac{dx_r}{dy_n}\frac{dy_n}{dx_s} = 0,$$

$$\frac{dx_r}{dy_1}\frac{dy_1}{dx_r} + \ldots + \frac{dx_r}{dy_n}\frac{dy_n}{dx_r} = 1.$$

These equations give

$$J\frac{dx_r}{dy_n} = \text{minor of } \frac{dy_n}{dx_r} \text{ in } J,$$

where J denotes the functional determinant or Jacobian

$$\frac{d(y_1, \ldots y_n)}{d(x_1, \ldots x_n)} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13).$$

Substituting the values of $\dfrac{dx_1}{dy_n}, \ldots \dfrac{dx_n}{dy_n}$ thus determined in (12), the equation (6) becomes

$$\frac{d(y_1, \ldots y_{n-1}, u)}{d(x_1, \ldots x_{n-1}, x_n)} = 0.$$

We have thus the theorem that if there exist between n functions $y_1, \ldots y_n$ of n independent variables $x_1, \ldots x_n$ an identical relation

$$\phi(y_1, \ldots y_n) = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14),$$

then the functional determinant (13) vanishes.

Conversely, we can show from the above process of integrating the equation (1) that if the Jacobian (12) vanishes, then there must exist an identical relation of the form (14). For the fact that (13) vanishes shows that the n equations of the type

$$X_1\frac{dy_r}{dx_1} + \ldots + X_n\frac{dy_r}{dx_n} = 0$$

are constant, *i.e.* that if the ratios

$$X_1 : X_2 : \ldots X_n$$

be determined from any $n-1$ of these equations, then the remaining equation will also be satisfied. But the n equations express precisely the fact that $y_1, \ldots y_n$ satisfy the equation

$$X_1 \frac{du}{dx_1} + \ldots + X_1 \frac{du}{dx_n} = 0.$$

Hence we must have $y_1 =$ some function of $u_1, \ldots u_{n-1}$, and so for $y_2, y_3 \ldots y_n$; *i.e.* there must be some relation between the functions $y_1, \ldots y_n$ of the form (14).

University, Melbourne,
August 3, 1877.

MATHEMATICAL NOTES.

On two related quadric functions.

Assume

$$\phi x = a^2 (c-x) - x (c^2 - b^2 - cx),$$

$$\psi x = b^2 (c-x) - x (c^2 - a^2 - cx),$$

then

$$\phi \left(\frac{a^2}{c-x}\right) = \frac{a^2}{(c-x)^2} \psi x,$$

$$\psi \left(\frac{b^2}{c-x}\right) = \frac{b^2}{(c-x)^2} \phi x.$$

In the first of these for x write $\dfrac{b^2}{c-x}$, then

$$\phi \left\{\frac{a^2 (c-x)}{c^2 - b^2 - cx}\right\} = \frac{a^2 (c-x)^2}{(c^2 - b^2 - cx)^2} \frac{b^2}{(c-x)^2} \phi x$$

$$= \frac{a^2 b}{(c^2 - b^2 - cx)^2} \phi (x).$$

A. Cayley.

On development in series.

Consider, in general, a function developable in a convergent series proceeding according to positive powers of the variables, and, for example, let

$$\frac{F(x)}{Ae^{\alpha x}+Be^{\beta x}+Ce^{\gamma x}}=a_0+a_1\frac{x}{1}+a_2\frac{x^2}{2!}+\ldots+a_n\frac{x^n}{n!}+\ldots,$$

or under a symbolic form

$$\frac{F(x)}{Ae^{\alpha x}+Be^{\beta x}+Ce^{\gamma x}}=e^{ax}.$$

Denote by $f(x)$ any other function whatever, and by h an increment of x; we have the symbolic formula

$$F(hf)=Ae^{ahf(x+\alpha h)}+Be^{ahf(x+\beta h)}+Ce^{ahf(x+\gamma h)},$$

in the development of which we are to replace

$$h^0f^0 \text{ by } f(x),$$

$$h^nf^n \text{ by } h^n\frac{d^nf(x)}{dx^n},$$

$$\{ahf(x+\alpha h)\}^n \text{ by } a_nh^n\frac{d^nf(x+\alpha h)}{dx^n}.$$

In fact, it is easy to see that this formula holds for $f(x)=Ge^{kx}$, whatever G and k may be, and therefore also for any function whatever ΣGe^{kx} of x.

We have, in particular, for $\dfrac{x}{e^x-1}=e^{Bx}$,

$$hf'(x)=e^{Bhf(x+h)}-e^{Bhf(x)},$$

which is Stirling's formula; and for $\dfrac{-2x}{e^x+1}=e^{Px}$, with $P_n=2(1-2^n)B_n$, we have the formula

$$-2f(x)=e^{Phf(x+h)}+e^{Phf(x)},$$

a formula due to Boole.

Let $\dfrac{2}{e^x+e^{-x}}=e^{Ex}$, E_n denoting an Eulerian number, then

$$2f(x)=e^{Ehf(x+h)}+e^{Ehf(x-h)},$$

and similarly for many other developments.

EDOUARD LUCAS.

Paris, *November*, 1877.

*On a Theorem due to Rodrigues.**

Let $\{(x+h)^2-1\}^i$ be the series of which the general term is $P_r h^r$, then

$$P_r = \frac{1}{\underline{|r}}\left(\frac{d}{dx}\right)^r (x^2-1)^i,$$

now

$$P_r = \text{coeff. of } h^r \text{ in } (x^2-1)^i\left(1+\frac{2x}{x^2-1}h+\frac{h^2}{x^2-1}\right)^i$$

$$= (x^2-1)^i \times \text{coeff. of } h^r \text{ in } \left(1+\frac{2x}{x^2-1}h+\frac{h^2}{x^2-1}\right)^i;$$

also

$$P_{i-m} = \text{coeff. of } h^{i-m} \text{ in } h^{2i}\left\{1+\frac{2x}{x^2-1}\,\frac{x^2-1}{h}+\frac{\left(\frac{x^2-1}{h}\right)^2}{x^2-1}\right\}^i$$

$$= \text{coeff. of } \frac{1}{h^{i+m}} \text{ in } \left\{1+\frac{2x}{x^2-1}\,\frac{x^2-1}{h}+\frac{\left(\frac{x^2-1}{h}\right)^2}{x^2-1}\right\}^i$$

$$= (x^2-1)^{i+m} \times \text{coeff. of } k^{i+m} \text{ in } \left(1+\frac{2x}{x^2-1}k+\frac{k^2}{x^2-1}\right)^i,$$

where $$k=\frac{x^2-1}{h}$$

$$= (x^2-1)^m P_{i+m};$$

therefore

$$\frac{1}{\underline{|i-m}}\left(\frac{d}{dx}\right)^{i-m}(x^2-1)^i = \frac{(x^2-1)^m}{\underline{|i+m}}\left(\frac{d}{dx}\right)^{i+m}(x^2-1)^i;$$

therefore

$$\left(\frac{d}{dx}\right)^{i-m}(x^2-1)^i = (x^2-1)^m\frac{\underline{|i-m}}{\underline{|i+m}}\left(\frac{d}{dx}\right)^{i+m}(x^2-1)^i,$$

which is the theorem in question.

W. H. H. HUDSON.

October 31, 1877.

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* See Ferrers's *Spherical Harmonics*, (1877), pp. 13–16, or Todhunter's *Laplace's Functions, &c.*, (1875), pp. 76–80.

Proof of the Principle of the Composition of Couples in Statics.

The following proof of the law of composition of couples seems simpler than that usually given, and will probably suggest Sir W. Thomson's proof of the composition of elementary rotations in a mass of fluid moving with rotation.

It is a well-known theorem in statics that three forces which act along the sides of a triangle and are proportional to them compound a couple whose axis is perpendicular to the plane of the triangle and whose moment is proportional to the area of the triangle. Conversely, if we take any tetrahedron and suppose the faces acted on by couples whose axes point inwards and whose moments are proportional to the faces in which they act, these couples may be replaced by forces round the sides of the triangle, which a little inspection will show mutually cancel each other. The couples are therefore in equilibrium.

This result is evidently equivalent to the law of the composition of couples.

C. NIVEN.

Expansion derived from Lagrange's Series.

Theorem. If $x = a + efx$ and $\varepsilon = \dfrac{e}{1+ne}$, n being any quantity, then

$$x = a + \varepsilon fa + \frac{\varepsilon^2}{1.2}\frac{d}{da}(na+fa)^2 + \frac{\varepsilon^3}{1.2.3}\frac{d^2}{da^2}(na+fa)^3 + \frac{\varepsilon^4}{4!}\frac{d^3}{da^3}(na+fa)^4 + \&c.,$$

in which, after the differentiations, a is to be put equal to zero, except under a functional sign.

Thus, performing the differentiations,

$$\frac{d}{da}(na+fa)^2 = 2(na+fa)(n+f'a),$$

$$\frac{d}{da}(na+fa)^3 = 3(na+fa)^2(n+f'a),$$

$$\frac{d^2}{da^2}(na+fa)^3 = 6(na+fa)(n+f'a)^2 + 3(na+fa)^2 f''a,$$

$$\&c. = \&c.$$

so that the theorem gives

$$x = a + \epsilon fa + \frac{\epsilon^2}{1.2}\{2fa(n+f'a)\}$$

$$+ \frac{\epsilon^3}{1.2.3}\{6fa(n+f'a)^2 + 3(fa)^2 f''a\} + \&c.$$

J. W. L. Glaisher.

Four Algebraical Theorems.

I. If $$A_n = \frac{x^2(x^2+2^2)\ldots\{x^2+(2n-2)^2\}}{(2n)!},$$

$$B_n = \frac{x(x^2+1^2)(x^2+3^2)\ldots\{x^2+(2n-1)^2\}}{(2n+1)!},$$

and $$A_0 = 1, \quad B_0 = x,$$

then $$A_n + \tfrac{1}{2}A_{n-1} + \frac{1.3}{2.4}A_{n-2} + \&c. = (2n+1)\frac{B_n}{x},$$

$$B_n + \tfrac{1}{2}B_{n-1} + \frac{1.3}{2.4}B_{n-2} + \&c. = (2n+2)\frac{A_{n+1}}{x}.$$

For example, $n = 2$,

$$\frac{x^2(x^2+2^2)}{4!} + \tfrac{1}{2}\frac{x^2}{2!} + \frac{1.3}{2.4} = 5\,\frac{(x^2+1^2)(x^2+3^2)}{5!},$$

$$\frac{x(x^2+1^2)(x^2+3^2)}{5!} + \tfrac{1}{2}\,\frac{x(x^2+1^2)}{3!} + \frac{1.3}{2.4}x = 6\,\frac{x(x^2+2^2)(x^2+4^2)}{6!}.$$

II. If $$A_n = \frac{n^{n-1}}{n!},$$

then $A_nA_1 + A_{n-1}A_2 + A_{n-2}A_3 \ldots + A_{\frac{1}{2}(n+2)}A_{\frac{1}{2}n} = \dfrac{n}{n+1}A_{n+1}$

if n be even, and

$$A_nA_1 + A_{n-1}A_2 + A_{n-2}A_3 \ldots + \tfrac{1}{2}A_{\frac{1}{2}(n+1)}A_{\frac{1}{2}(n+1)} = \frac{n}{n+1}A_{n+1},$$

(the last term of the series having the factor $\frac{1}{2}$), if n be uneven.
For example, $n = 6$, the first equation gives

$$\frac{6^5}{6!} + \frac{5^4}{5!}\cdot\frac{2^1}{2!} + \frac{4^3}{4!}\cdot\frac{3^2}{3!} = \frac{6}{7}\cdot\frac{7^6}{7!},$$

and, $n=7$, the second equation gives

$$\frac{7^6}{7!}+\frac{6^5}{6!}\cdot\frac{2^1}{2!}+\frac{5^4}{5!}\cdot\frac{3^2}{3!}+\tfrac{1}{2}\cdot\frac{4^3}{4!}\cdot\frac{4^3}{4!}=\frac{7}{8}\cdot\frac{8^7}{8!}.$$

III. If S_n denote the sum of the products of the quantities

$$\frac{1}{1+1+1^2},\ \frac{1}{1+2+2^2},\ \frac{1}{1+3+3^2},\ \dots\ ad.\ inf.,$$

taken n together, then

$$S_1+S_2-S_3-S_4+S_5+S_6-S_7-S_8+\&c.\ ad.\ inf.=1,$$

the terms being alternately positive and negative in pairs.

IV. If

$$\frac{\sin(b-c)}{b-c}\cdot\frac{\sin(c-a)}{c-a}\cdot\frac{\sin(a-b)}{a-b}=1+A_2+A_4+A_6+A_8+\&c.,$$

where A_n denotes the terms of n dimensions in a, b, c, then

$$A_4,\ A_{10},\ A_{16}\dots A_{6n-2},\ \dots$$

all contain the algebraical factor $(a^2+b^2+c^2-bc-ca-ab)^2$, and

$$A_2,\ A_8,\ A_{14}\dots A_{6n+2},\ \dots$$

all contain the algebraical factor $a^2+b^2+c^2-bc-ca-ab$.

J. W. L. GLAISHER.

Elementary Proof of a Theorem in Functional Determinants.

The theorem, due to Jacobi, is as follows:

Let there be n functions $\phi_1, \dots\phi_n$ of n independent variables $x_1, \dots x_n$; then if there be an identical relation

$$F(\phi_1, \dots\phi_n)=0 \quad\dots\dots\dots\dots\dots\dots\dots(1)$$

between the function ϕ, the functional determinant

$$\frac{d(\phi_1, \dots\phi_n)}{d(x_1, \dots x_n)}$$

will vanish; and, conversely, if the functional determinant vanishes, then there will be an identical relation of the form (1).

The theorem is well known and many different proofs have been given. Baltzer,* Boole,† and Laurent‡ make use

* *Theorie und Anwendung der Determinanten*, 3rd edition p. 131.
† *Differential Equations*, sup. vol., p. 56.
‡ *Mécanique Rationelle*, p. 330.

of differentials and in a proof given by Brioschi,* the method of induction is used; the latter proof is, so far as I can remember, very similar to the one originally given by Jacobi.

The following proof is direct and depends only on th use of differential coefficients. It is the extension of the proof given by Boole† in the case of two functions.

The proof of the first part is obvious, viz. from (1) we have by differentiation n equations of the form

$$\frac{dF}{d\phi_1}\frac{d\phi_1}{dx_r}+\ldots+\frac{dF}{d\phi_n}\frac{d\phi_n}{dx_r}=0,$$

and, eliminating $\dfrac{dF}{d\phi_1},\ldots\dfrac{dF}{d\phi_n}$ from these equations, we get

$$\frac{d(\phi_1,\ldots\phi_n)}{d(x_1,\ldots x_n)}=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The converse is proved as follows:

The functions $\phi_1,\ldots\phi_{n-1}$ are mutually independent, or else the proposition to be proved is granted. We can then express ϕ_n in terms of $\phi_1,\ldots\phi_{n-1}$ and ψ_n, where ψ_n is any function which is independent of $\phi_1,\ldots\phi_{n-1}$. If this be done we have by a known theorem

$$\frac{d(\phi_1,\ldots\phi_n)}{d(x_1,\ldots x_n)}=\frac{d(\phi_1,\ldots\phi_{n-1},\ \phi_n)}{d(\phi_1,\ldots\phi_{n-1},\ \psi_n)}\cdot\frac{d(\phi_1,\ldots\phi_{n-1},\psi_n)}{d(x_1,\ldots x_{n-1},\ x_n)}$$

$$=\frac{d\phi_n}{d\psi_n}\cdot\frac{d(\phi_1,\ldots\phi_{n-1},\ \psi_n)}{d(x_1,\ldots x_{n-1},\ x_n)}.$$

Now $\phi_1,\ldots\phi_{n-1},\ \psi_n$ are mutually independent, so that

$$\frac{d(\phi_1,\ldots\phi_{n-1},\ \psi_n)}{d(x_1,\ldots x_{n-1},\ x_n)} \text{ not } =0.$$

Hence, by (2), we have

$$\frac{d\phi_n}{d\psi_n}=0,$$

which proves the proposition. E. J. NANSON.

University, Melbourne,
August 3, 1877.

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* *Théorie des Déterminants*, p. 122.
† *Differential Equations*, p. 24.

Proof of the formula for $1^3+2^3+3^3\ldots+n^3$*.

The sum of the cubes of the first n natural numbers may be found as follows:

Let AB, BC, CD, &c. (fig. 7) represent 1, 2, 3, &c. Draw BE, CF, DG, &c. representing 1^2, 2^2, 3^2, &c. Then the rectangles AE, BF, CG, &c. will represent 1^3, 2^3, 3^3, &c. Draw AEK at 45° to AD, and suppose CG is the n^{th} rectangle, *i.e.*, $=n^3$. Then $GK=n^2-\frac{1}{2}n(n+1)=Hh$; therefore the figure $HKGL$ will fit into the position $HKlh$. Similarly, $EHFM$ will fit into position $EHhe$, and so make up the square AK, or $S_{n,3}=\{\frac{1}{2}n(n+1)\}^2$.

The squares may also be summed in the following simple way:

$$n^2=1+3+5+\ldots+(2n-1),$$

$$(n-1)^2=1+3+5+\ldots+(2n-3),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$2^2=1+3,$$

$$1^2=1;$$

therefore $S_{n,2}=n\ +3(n-1)+5(n-2)+\ldots+(2n-1),$

and $2S_{n,2}=2n^2+2(n-1)^2+2(n-2)^2+\ldots+2.$

Adding, we get

$$3S_{n,2}=(2n+1)\{n+(n-1)+(n-2)+\ldots+1\},$$

whence $S_{n,2}=\dfrac{n(n+1)(2n+1)}{1.2.3}.$

J. M. Croker.

Expansion of products of cosines and sines.

The following note contains the expansion of

$$\cos\theta_1\cos\theta_2\ldots\cos\theta_n \text{ and of } \sin\theta_1\sin\theta_2\ldots\sin\theta_n$$

in terms of the sums of certain cosines and sines.

Let $x_1=\cos\theta_1+i\sin\theta_1$, $x_2=\cos\theta_2+i\sin\theta_2$, &c., and let $x_1x_2\ldots x_n$ be denoted by $P^n(x)$, $\cos\theta_1\cos\theta_2\ldots\cos\theta_n$ by $P^n(\cos\theta)$, $\sin\theta_1\sin\theta_2\ldots\sin\theta_n$ by $P^n(\sin\theta)$, and $\theta_1+\theta_2+\ldots+\theta_n$ by $\Sigma_n(\theta)$, then we have

$$P^n(x)=\cos\Sigma_n\theta+i\sin\Sigma_n\theta,$$

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* See p. 60.

therefore $$\cos \Sigma_n\theta = \tfrac{1}{2}\left\{P^n(x) + \frac{1}{P^n(x)}\right\},$$
$$\sin \Sigma_n\theta = \tfrac{1}{2}\left\{P^n(x) - \frac{1}{P^n(x)}\right\};$$
also $$2^n P^n(\cos\theta) = P^n\left(x + \frac{1}{x}\right).$$

By the symbol $\Sigma\phi[A_r\theta]$ we mean the sum of the different values taken by $\phi[A_r\theta]$, when for $A_r\theta$ is substituted the sum of any r angles, until the whole number of possible combinations is exhausted, supposing them all unlike.

Thus, if $r=2$, $n=3$,

$$\begin{aligned}\Sigma \cos\{\Sigma_n(\theta) - 2A_r\theta\} &= \cos\{\theta_1+\theta_2+\theta_3 - 2(\theta_1+\theta_2)\}\\ &+ \cos(\theta_1+\theta_2+\theta_3 - 2(\theta_1+\theta_3)\} + \cos(\theta_1+\theta_2+\theta_3 - 2(\theta_2+\theta_3)\}\\ &= \cos(\theta_3-\theta_1-\theta_2) + \cos(\theta_2-\theta_1-\theta_3) + \cos(\theta_1-\theta_2-\theta_3).\end{aligned}$$

Now, in expanding $P^n\left(x+\frac{1}{x}\right)$, we notice

$$\text{first term } P^n(x) + \text{last term } \frac{1}{P^n(x)} = 2\cos\Sigma\theta,$$

$$\text{second term, } P^n(x)\left\{\frac{1}{x_1^2} + \frac{1}{x_2^2} + \ldots + \frac{1}{x_n^2}\right\}$$

$$+ \text{ last term but one, } \frac{1}{P^n(x)}\{x_1^2+x_2^2+\ldots+x_n^2\} = 2\Sigma\cos\{\Sigma\theta - 2A_1\theta\}.$$

So, adding the third term and the last term but two, we have sum $= 2\Sigma\cos(\Sigma\theta - 2A_2\theta)$.

If n be odd there will be no singular term in the middle.

If n be even, say $2q$, there will be a middle term.

$$\Sigma\left\{\frac{x_1x_2\ldots x_q}{x_{q+1}x_{q+2}\ldots x_{2q}} + \frac{x_{q+1}x_{q+2}\ldots x_{2q}}{x_1x_2\ldots x_q}\right\}\ldots,$$

and so on with other elements which may be coupled in the same way.

Thus, taking the whole twice over, the middle term is $\Sigma\cos(\Sigma_n\theta - 2A_q\theta)$.

We are able to say therefore that generally

$$\begin{aligned}2^{n-1}P^n(\cos\theta) = \cos A_n\theta &\\ + \Sigma\cos(A_n\theta - 2A_1\theta)&\\ + \Sigma\cos(A_n\theta - 2A_2\theta)&\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots&\\ + \Sigma\cos(A_n\theta - 2A_r\theta).&\end{aligned}$$

If n be odd $=2p+1$, last terms $=\Sigma\cos(A_n\theta-2A_p\theta)$.
If n be even $=2q$, last terms $=\frac{1}{2}\Sigma\cos(A_n\theta-2A_q\theta)$.
Practically this set of terms splits into two equal parts so that there is no fraction.

Examples:

$$2\cos\alpha\cos\beta=\cos(\alpha+\beta)+\tfrac{1}{2}\cos(\alpha-\beta)+\tfrac{1}{2}\cos(\beta-\alpha)$$
$$=\cos(\alpha+\beta)+\cos(\alpha-\beta),$$
$$4\cos\alpha\cos\beta\cos\gamma=\cos(\alpha+\beta+\gamma)$$
$$+\cos(\alpha+\beta-\gamma)+\cos(\alpha+\gamma-\beta)+\cos(\beta+\gamma-\alpha).$$

Again, with the product of sines.

If n be odd $=2p+1$,

$$2^{n-1}(-1)^{\frac{1}{2}(n-1)}P^n(\sin\theta)=\sin A_n\theta-\Sigma\sin(A_n\theta-2A_1\theta)$$
$$+\Sigma\sin(A_n\theta-2A_2\theta)+\ldots(-1)^p\Sigma\sin(A_n\theta-2A_p\theta).$$

If n be even $=2q$,

$$2^{n-1}(-1)^{\frac{1}{2}n}P^n(\sin\theta)=\cos A_n\theta-\Sigma\cos(A_n\theta-2A_1\theta)$$
$$+\Sigma\cos(A_n\theta-2A_2\theta)+\ldots\tfrac{1}{2}(-1)^q\Sigma\cos(A_n\theta-2A_q\theta).$$

Practically, this last set of terms splits into two equal parts.

Examples:

$$2\sin\alpha\sin\beta=\tfrac{1}{2}\cos(\alpha-\beta)+\tfrac{1}{2}\cos(\beta-\alpha)-\cos(\alpha+\beta)$$
$$=\cos(\alpha-\beta)-\cos(\alpha+\beta),$$
$$4\sin\alpha\sin\beta\sin\gamma=\sin(\alpha+\beta-\gamma)+\sin(\alpha+\gamma-\beta)$$
$$+\sin(\beta+\gamma-\alpha)-\sin(\alpha+\beta+\gamma).$$

R. VERDON.

A Trigonometrical Identity.

$$\cos(b-c)\cos(b+c+d)+\cos a\cos(a+d)$$
$$=\cos(c-a)\cos(c+a+d)+\cos b\cos(b+d)$$
$$=\cos(a-b)\cos(a+b+d)+\cos c\cos(c+d)$$
$$=\cos a\cos(a+d)+\cos b\cos(b+d)+\cos c\cos(c+d)-\cos d.$$

A. CAYLEY.

Extract of a Letter from Prof. Cayley.

"I wish to construct a correspondence such as

$$(x+iy)^3+(x+iy)=X+iY,$$

or, say, for greater convenience

$$4\,(x+iy)^3-3\,(x+iy)=X+iY,$$

viz. if $\qquad x+iy \;=\cos u,$

then $\qquad X+iY=\cos 3u.$

Suppose $3u_0$ is a value of $3u$ corresponding to a given value of $X+iY$, then the three values of $x+iy$ are of course $\cos u_0$, $\cos\left(u_0\pm\frac{2\pi}{3}\right)$, but I am afraid the calculation of u_0, even with cosh and sinh tables would be very laborious. Writing $X+iY=R\,(\cos\Theta+i\sin\Theta)$, the intervals for Θ might be 5°, 10° or even 15°, those of R, say 0·1 from 0 to 2, and then 0·5 up to 4 or 5; and 2 places of decimals would be quite sufficient; but even this would probably involve a great mass of calculation.

It has occurred to me that perhaps a geometrical solution might be found for the equation $X+iY=\cos 3u$."

A. CAYLEY.

October 31, 1877.

THEOREM IN KINEMATICS.

By *C. Leudesdorf, M.A.*

LET A, B, C, P (fig. 8) be four points rigidly connected together and moving together in any way in a plane. Let $BC=a$, $CA=b$, $AB=c$, $PA=a'$, $PB=b'$, $PC=c'$, $PAB=\alpha$, $PBC=\beta$, $PCA=\gamma$, also let A be (x_1y_1), $B\,(x_2y_2)$, $C\,(x_3y_3)$, $P\,(xy)$, and let AB make an angle θ with Ox, at any moment. Then

$$x_1=x-a'\cos(\theta+\alpha),\; x_2=x+b'\cos(\theta+\beta-B),\; x_3=x-c'\cos(\theta+\gamma+A),$$

$$y_1=y-a'\sin(\theta+\alpha),\; y_2=y+b'\sin(\theta+\beta-B),\; y_3=y-c'\sin(\theta+\gamma+A);$$

therefore

$$\tfrac{1}{2}\,(x_1dy_1-y_1dx_1)=\tfrac{1}{2}\,(xdy-ydx)-\tfrac{1}{2}a'\left\{\cos(\theta+\alpha)\,(yd\theta+dx)\right.$$
$$\left.+\sin(\theta+\alpha)\,(yd\theta-dx)\right\}+\tfrac{1}{2}a'^2d\theta \;\ldots\ldots\ldots\ldots\ldots(1),$$

$$\tfrac{1}{2}\,(x_2dy_2-y_2dx_2)=\tfrac{1}{2}\,(xdy-ydx)+\tfrac{1}{2}b'\left\{\cos(\theta+\beta-B)(yd\theta+dx)\right.$$
$$\left.+\sin(\theta+\beta-B)\,(yd\theta-dx)\right\}+\tfrac{1}{2}b'^2d\theta \;\ldots\ldots\ldots(2),$$

$$\tfrac{1}{2}\,(x_3dy_3-y_3dx_3)=\tfrac{1}{2}\,(xdy-ydx)-\tfrac{1}{2}c'\left\{\cos(\theta+\gamma+A)(yd\theta+dx)\right.$$
$$\left.+\sin(\theta+\gamma+A)\,(yd\theta-dx)\right\}+\tfrac{1}{2}c'^2d\theta \;\ldots\ldots\ldots(3).$$

Multiplying (1), (2), (3) by λ, μ, ν respectively and adding, we have, if (A), (B), (C), (P) denote the areas of the curves traced out by A, B, C, P,

$$\lambda d(A) + \mu d(B) + \nu d(C)$$
$$= (\lambda + \mu + \nu)\, d(P) + \tfrac{1}{2}(\lambda a'^2 + \mu b'^2 + \nu c'^2)\, d\theta \ldots\ldots (4),$$

provided only that

$$\left.\begin{array}{l} -\lambda a' \cos\alpha + \mu b' \cos(\beta - B) + \nu c' \cos(\gamma + A) = 0 \\ -\lambda a' \sin\alpha + \mu b' \sin(\beta - B) + \nu c' \sin(\gamma + A) = 0 \end{array}\right\};$$

whence

$$\begin{aligned} \lambda a' : \mu b' : \nu c' &= \sin(\beta - \gamma + C) : \sin(\gamma - \alpha + A) : \sin(\alpha - \beta + B) \\ &= \sin BPC \qquad : \sin CPA \qquad : \sin APB. \end{aligned}$$

Substituting in (4), and assuming that the curves traced out by the four points are closed, we have, by integrating for θ from 0 to 2π,

$$\frac{\sin BPC}{a'}(A) + \frac{\sin CPA}{b'}(B) + \frac{\sin APB}{c'}(C)$$
$$= \left(\frac{\sin BPC}{a'} + \frac{\sin CPA}{b'} + \frac{\sin APB}{c'}\right)(P)$$
$$+ \pi\,(a' \sin BPC + b' \sin CPA + c' \sin APB) \ldots\ldots\ldots\ldots (5).$$

The coefficient of (P) is equal to $\dfrac{2\Delta ABC}{a'b'c'}$; that of π

$$= \frac{2}{a'b'c'}(a'^2 x + b'^2 y + c'^2 z),$$

if x, y, z be the triangular coordinates of P referred to the triangle ABC,

$$= \frac{2}{a'b'c'}\left[\{a^2 yz + b^2 z(x-1) + c^2 (x-1) y\}\, x + \ldots + \ldots\right]$$
$$= \frac{2}{a'b'c'}(a^2 yz + b^2 zx + c^2 xy),$$

so that (5) reduces to

$$x(A) + y(B) + z(C) = (P) + \pi\,(a^2 yz + b^2 zx + c^2 xy),$$

or $$(P) = x(A) + y(B) + z(C)$$

$+\ \pi$ (square of tangent from P to circle round ABC).

If then A and B be made to move on given closed curves, the above formula connects the areas of the curves traced out by the two carried points C and P.

We may notice a few particular cases.

If we suppose P to lie on AB, and to divide it in the ratio $c : c'$, we have $z=0$, $x : y : 1 = c' : c : c+c'$, and the formula reduces to

$$(P) = \frac{c(B) + c'(A)}{c+c'} - \pi cc',$$

i.e. we have Holditch's theorem, as given in Williamson's *Integral Calculus*, p. 200.

If A and B be made to move on the same curve (or on curves of equal area) and if C then move on the same curve (or on one of equal area), then

$$(P) = (A) + \pi \text{ (square of tangent from } P \text{ to circle } ABC).$$

If A and B move on fixed circles of radii r, s the loci of C and P are three-bar curves, whose areas are connected by the equation

$$(P) = z(C) + \pi (xr^2 + ys^2 - a^2yz - b^2zx - c^2xy),$$

and in the particular case when $r=b$, $s=a$,

$$\begin{aligned}(P) &= z(C) + \pi (b^2x + a^2y - a^2yz - b^2zx - c^2xy) \\ &= z(C) + \pi . PC^2.\end{aligned}$$

TRANSACTIONS OF SOCIETIES.

London Mathematical Society.

Thursday, November 8th, 1877.—Lord Rayleigh, F.R.S., *President*, in the chair.—The following were elected to form the Council during the Session:—*President:* Lord Rayleigh, F.R.S. *Vice-Presidents:* Prof. J. Clerk Maxwell, F.R.S., Mr. C. W. Merrifield, F.R.S., Prof. H. J. S. Smith, F.R.S. *Treasurer:* Mr. S. Roberts, M.A. *Hon. Secretaries:* Messrs. M. Jenkins, M.A., and R. Tucker, M.A. Other members, Prof. Cayley, F.R.S., Mr. T. Cotterill, M.A., Mr. J. W. L. Glaisher, F.R.S., Mr. H. Hart, M.A., Dr. Henrici, F.R.S., Dr. Hirst, F.R.S., Mr. Kempe, B.A., Dr. Spottiswoode, F.R.S., Mr. J. J. Walker, M.A. The Rev. W. Ellis, B.A., Caius College, Cambridge, was proposed for election. Prof. Cayley made two communications, on the function $\phi(x) = \dfrac{ax+b}{cx+d}$ and on the theta functions. In the first of these papers the value obtained for the nth function is substantially of the same form as that found long ago by Babbage, but is more compendiously expressed; the result is

$$\phi^n(x) = \frac{(\lambda^{n+1}-1)(ax+b) + (\lambda^n - \lambda)(-dx+b)}{(\lambda^{n+1}-1)(cx+d) + (\lambda^n-\lambda)(cx-a)},$$

where $\frac{(\lambda+1)^2}{\lambda} = \frac{(a+d)^2}{ad-bc}$. It was arrived at in a simple manner by means of the identity

$$\begin{vmatrix} M-a, & b \\ c, & M-d \end{vmatrix} = 0, \text{ or } M^2 - (a+d)M + (ad-bc) = 0,$$

satisfied by the quadric matrix $M = \begin{vmatrix} a, & b \\ c, & d \end{vmatrix}$.

The second communication consisted of an account of researches upon the double theta functions, on which Prof. Cayley is engaged; as an introduction, he establishes in a strictly analogous manner the theory of the single theta functions. Mr. Tucker read a portion of a paper by Mr. Hugh MacColl (communicated by Prof. Crofton, F.R.S.) entitled "The Calculus of Equivalent Statements." A short account of this analytical method has been given in the July and November numbers (1877) of the *Educational Times*, under the name of Symbolical Language. The chief use at present made of it is to determine the new limits of integration when we change the order of integration or the variables in a multiple integral, and also to determine the limits of integration in questions relating to probability. This object, the writer asserts, it will accomplish with perfect certainty, and by a process almost as simple and mechanical as the ordinary operations of elementary algebra.—The President read a paper on Progressive Waves. It has often been remarked that when a group of waves advance into still water the velocity of the group is less than that of the individual waves of which it is composed; the waves appear to advance through the group, dying away as they approach its anterior limit. This phenomenon seems to have been first explained by Prof. Stokes, who regarded the group as formed by the superposition of two infinite trains of waves of equal amplitudes and of nearly equal wave-lengths advancing in the same direction. The writer's attention was called to the subject about two years since by Mr. Froude, and the same explanation then occurred to him independently. In his work on "The Theory of Sound" (§ 191), he has considered the question more generally. In a paper read at the Plymouth meeting of the British Association (afterwards printed in *Nature*), Prof. Osborne Reynolds gave a dynamical explanation of the fact that a group of deep-water waves advances with only half the rapidity of the individual waves. Another phenomenon (also mentioned to the author by Mr. Froude) was also discussed as admitting of a similar explanation to that given in the present paper. A steam launch moving quickly through the water is accompanied by a peculiar system of diverging waves, of which the most striking feature is the obliquity of the line containing the greatest elevation of successive waves to the wave-fronts. This wave-pattern may be explained by the superposition of two (or more) infinite trains of waves, of of slightly differing wave-lengths, whose direction and velocity of propagation are so related in each case that there is no change of position relatively to the boat. The mode of composition will be best understood by drawing on paper two sets of parallel and equidistant lines, subject to the above conditions, to represent the crests of the component trains. In the case of two trains of slightly different wave-lengths, it may be proved that the tangent of the angle between the line of maxima and the wave-fronts is half the tangent of the angle between the wave-fronts and the boat's course.—Prof. Clifford, F.R.S., communicated three notes. (1) On the triple generation of three-bar curves. *If one of the three-bar systems is a crossed rhomboid, the other two are kites.* This follows from the known fact that the path of the moving point in both these cases is the inverse of a conic. But it is also intuitively obvious as soon as the figure is drawn, and thus supplies an elementary proof that the path *is* the inverse of a conic in the case of a kite, which is not otherwise easy to get. (2) On the mass-centre of an octahedron. The construction was suggested by Dr. Sylvester's construction for the mass-centre of a tetrahedral frustum. (3) On vortex-motion. The problem solved by Stokes may, as a general question of analysis, be stated as follows: Given the expansion and the rotation at every point of a moving substance, it is required to find the velocity at every point. The solution was exhibited in a very simple form.

R. TUCKER, M.A., *Hon. Sec.*

ON EQUIVALENT LENSES.

By *R. Pendlebury, M.A.*

A LENS is said to be equivalent to a system of n lenses on the same axis, when being placed in the position of the first lens it produces the same deviation on a given ray as the system. The general formula for the focal length of a lens equivalent to n lenses of focal lengths $f_1, f_2 \ldots f_n$, and separated by distances $a_1, a_2 \ldots a_{n-1}$ can be readily found.

A ray passing through a lens at a distance y from the axis suffers a deviation $\frac{y}{f}$ from the axis. Suppose a ray parallel to the axis to be incident on the first lens of the system at a distance y_1 from the axis. Let δ_1 be the deviation after passing through the first lens, y_2 the distance from the axis at which the ray cuts the second lens, δ_2 its deviation after the second refraction, and so on. Then we get the following system of equations $\left(\text{writing } k_1, k_2 \ldots k_n \text{ for } \frac{1}{f_1}, \frac{1}{f_2} \ldots \frac{1}{f_n}\right)$,

$$\delta_1 = k_1 y_1,$$

$$y_2 = a_1 \delta_1 + y_1,$$

$$\delta_2 = k_2 y_2 + \delta_1,$$

$$y_3 = a_2 \delta_2 + y_2,$$

&c.

$$\delta_n = k_n y_n + \delta_{n-1}.$$

Forming the continued fraction

$$\frac{y_1}{1+} \frac{1}{k_1 +} \frac{1}{a_1 +} \frac{1}{k_2 +} \frac{1}{a_2 +} \ldots + \frac{1}{k_n},$$

it is clear that the numerator of the last convergent is δ_n. But if F_n be the focal length of the equivalent lens $\frac{y_1}{F} = \delta_n$.

Hence $\frac{1}{F_n}$ is equal to the numerator of the last convergent to the continued fraction

$$\frac{1}{1 +} \frac{1}{k_1 +} \frac{1}{a_1 +} \frac{1}{k_2 +} \ldots + \frac{1}{k_n}.$$

If the ray, instead of being parallel to the axis before meeting the first lens, cuts the axis at a distance d from that lens, we get the formula

$$\frac{1}{F}+\frac{1}{d}=\text{numerator of } \frac{1}{1+}\,\frac{1}{k+\theta+}\,\frac{1}{a_1+}\cdots+\frac{1}{k_n} \text{ where } \theta=\frac{1}{d}.$$

The expression may be obtained in the shape of a determinant. Solving the system of linear equations above,

$$\frac{1}{F_n}=-\begin{vmatrix} -k_n, -1, & 0, & 0, \ldots & 0, & 0 \\ 1, -a_{n-1}, -1, & 0, & \ldots & 0, & 0 \\ 0, & 0, -k_{n-1}, -1, & \ldots & 0, & 0 \\ 0, & 0, & 1, -a_{n-2} \ldots & 0, & 0 \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ 0, & 0, & 0, & 0, \ldots & -a_1, & -1 \\ 0, & 0, & 0, & 0, \ldots & 1, & -k_1 \end{vmatrix} \; (2n-1) \text{ terms.}$$

For example in the case of two lenses

$$\frac{1}{F_2}=-\begin{vmatrix} -k_2, & -1, & 0 \\ 1, & -a_1, & -1 \\ 0, & 1, & -k_1 \end{vmatrix}\equiv\frac{1}{f_1}+\frac{1}{f_2}+\frac{a_1}{f_1f_2};$$

In the case of three lenses

$$\frac{1}{F_3}=-\begin{vmatrix} -k_3, & -1, & 0, & 0, & 0 \\ 1, & -a_2, & -1, & 0, & 0 \\ 0, & 1, & -k_2, & -1, & 0 \\ 0, & 0, & 1, & -a_1, & -1 \\ 0, & 0, & 0, & 1, & -k_1 \end{vmatrix}$$

$$\equiv\frac{1}{f_1}+\frac{1}{f_2}+\frac{1}{f_3}+\frac{a_1}{f_1}\left(\frac{1}{f_2}+\frac{1}{f_3}\right)+\frac{a_2}{f_3}\left(\frac{1}{f_1}+\frac{1}{f_2}\right)+\frac{a_1a_2}{f_1f_2f_3}.$$

Another formula serves to connect the values of two consecutive terms of the series F_1, F_2...F_n, and gives an easy method of calculating them. Writing K_n for $\frac{1}{F_n}$, we have

$$K_n=(1+a_{n-1}k_n)\,K_{n-1}+k_n\frac{dK_{n-1}}{dk_{n-1}}.$$

Thus, for example,

$$\frac{1}{F_4} = (1 + a_3k_4)\,\{k_1+k_2+k_3+a_2k_3\,(k_1+k_2) + a_1k_1\,(k_2+k_3) + a_1a_2k_1k_2k_3\}$$

$$+ k_4\,\{1 + a_2\,(k_1 + k_2) + a_1k_1 + a_1a_2k_1k_2\}$$

$$= \frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3} + \frac{1}{f_4} + \frac{a_1}{f_1}\left(\frac{1}{f_2} + \frac{1}{f_3} + \frac{1}{f_4}\right) + a_2\left(\frac{1}{f_1} + \frac{1}{f_2}\right)\left(\frac{1}{f_3} + \frac{1}{f_4}\right)$$

$$+ \frac{a_3}{f_4}\left(\frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3}\right) + \frac{a_2a_3}{f_3f_4}\left(\frac{1}{f_1} + \frac{1}{f_2}\right) + \frac{a_3a_1}{f_4f_1}\left(\frac{1}{f_2} + \frac{1}{f_3}\right)$$

$$+ \frac{a_1a_2}{f_1f_2}\left(\frac{1}{f_3} + \frac{1}{f_4}\right) + \frac{a_1a_2a_3}{f_1f_2f_3f_4}.$$

ON SPHERICAL HARMONICS.

By *W. D. Niven, M.A.*, Trinity College, Cambridge.

§ 1. If V be the values at the point x, y, z of a function satisfying Laplace's equation, and if V_0 be the value at the origin of coordinates, then

$$V = e^{x\frac{d}{dx} + y\frac{d}{dy} + z\frac{d}{dz}}\,V_0 \qquad (1),$$

or

$$V_0 + x\left(\frac{dV}{dx}\right)_0 + y\left(\frac{dV}{dy}\right)_0 + z\left(\frac{dV}{dz}\right)_0 + \text{etc.} \qquad (2).$$

It was pointed out in a former paper in this Journal, that the peculiarity of this expansion is, that any set of homogeneous terms in x, y, z, say the $(i+1)^{\text{th}}$, viz.

$$\frac{1}{\lfloor i}\left(x\frac{d}{dx} + y\frac{d}{dy} + z\frac{d}{dz}\right)^i V_0 \qquad (3),$$

of itself satisfies Laplace's equation. The expression (3) is therefore a solid harmonic of the i^{th} degree. Moreover, since all the differentiations of the i^{th} degree are involved in it, and there are $2i+1$ independent ones, we may regard (3) as a compendious form of the aforesaid harmonic of the most general character.

§ 2. We shall now employ this form in the proof of several theorems, and as a preliminary step we shall find the value of

$$\iint e^{ax+by+cz}\,dS \qquad (4),$$

the integrations being taken over the surface of a sphere whose radius is R and centre the origin. We may clearly in that case, by change of axes, throw the integral into the form

$$\int_{-R}^{R} e^{\xi\sqrt{(a^2+b^2+c^2)}}\, 2\pi R d\xi,$$

the value of which is

$$2\pi R^2 \frac{e^{R\sqrt{(a^2+b^2+c^2)}} - e^{-R\sqrt{(a^2+b^2+c^2)}}}{R\sqrt{(a^2+b^2+c^2)}},$$

or $4\pi R^2 \left\{1 + \frac{1}{\lfloor 3} R^2 (a^2+b^2+c^2) + \ldots + \frac{1}{\lfloor 2i+1} R^{2i} (a^2+b^2+c^2)^i + \ldots\right\}$

$$\ldots\ldots\ldots\ldots\ldots\ldots \text{(5)}.$$

§ 3. The integral

$$\iint e^{x\frac{d}{dx}+y\frac{d}{dy}+z\frac{d}{dz}} V_0 dS$$

taken over the sphere is thus seen to be

$$4\pi R^2 V_0.$$

§ 4. Let V' be a second expression similar to V, and let us put

$$V' = e^{x\frac{d'}{dx}+y\frac{d'}{dy}+z\frac{d'}{dz}} V_0'.$$

Then $\iint VV'dS = \iint e^{x\frac{d}{dx}+y\frac{d}{dy}+z\frac{d}{dz}} V_0 e^{x\frac{d'}{dx}+y\frac{d'}{dy}+z\frac{d'}{dz}} V_0' dS$

$$= \iint e^{x\left(\frac{d}{dx}+\frac{d'}{dx}\right)+y\left(\frac{d}{dy}+\frac{d'}{dy}\right)+z\left(\frac{d}{dz}+\frac{d'}{dz}\right)} V_0 V_0' dS,$$

where the differentiators $\frac{d}{dx}$, $\frac{d}{dy}$, $\frac{d}{dz}$ operate on V only, and $\frac{d'}{dx}$, $\frac{d'}{dy}$, $\frac{d'}{dz}$ on V' only.

If we assume that the integral over the sphere of the product of two harmonics of unequal degree is zero, the two expressions for $\iint VV'dS$ lead by expansions of the exponential respectively to

$$\Sigma \iint \frac{\left(x\frac{d}{dx}+y\frac{d}{dy}+z\frac{d}{dz}\right)^i V_0}{\lfloor i} \cdot \frac{\left(x\frac{d'}{dx}+y\frac{d'}{dy}+z\frac{d'}{dz}\right)^i V_0' dS}{\lfloor i},$$

and, since $\left(\frac{d}{dx}+\frac{d'}{dx}\right)^2+\left(\frac{d}{dy}+\frac{d'}{dy}\right)^2+\left(\frac{d}{dz}+\frac{d'}{dz}\right)^2$

is really equivalent to

$$2\left(\frac{d}{dx}\frac{d'}{dx}+\frac{d}{dy}\frac{d'}{dy}+\frac{d}{dz}\frac{d'}{dz}\right),$$

by § 5, to

$$4\pi R^2\Sigma\,\frac{2^i}{\lfloor 2i+1}\,R^{2i}\left(\frac{d}{dx}\frac{d'}{dx}+\frac{d}{dy}\frac{d'}{dy}+\frac{d}{dz}\frac{d'}{dz}\right)^i V_0V_0'.$$

Since the corresponding terms of the same dimensions in R must be equal, we must have

$$\iint\frac{\left(x\frac{d}{dx}+y\frac{d}{dy}+z\frac{d}{dz}\right)^iV_0}{\lfloor i}\cdot\frac{\left(x\frac{d'}{dx}+y\frac{d'}{dy}+z\frac{d'}{dz}\right)^iV_0'}{\lfloor i}\,dS$$

$$=\frac{4\pi 2^iR^{2i+2}}{\lfloor 2i+1}\left(\frac{d}{dx}\frac{d'}{dx}+\frac{d}{dy}\frac{d'}{dy}+\frac{d}{dz}\frac{d'}{dz}\right)V_0V_0'.$$

§ 5. The simplest example of this theorem will be its application in finding $\iint Q_i^2dS$, where Q_i is a zonal harmonic having its pole in the axis of z. We shall suppose each of the expressions inside the integral to be equal to Q_ir^i, that is, we shall take

$$\frac{\left(x\frac{d}{dx}+y\frac{d}{dy}+z\frac{d}{dz}\right)^iV_0}{\lfloor i}$$

$$=\frac{\lfloor 2i}{2^i\lfloor i\lfloor i}z^i-\frac{\lfloor 2i-2}{2^i\lfloor i-1\lfloor i-2}z^{i-2}(x^2+y^2+z^2)+\ldots,$$

and

$$\frac{\left(x\frac{d'}{dx}+y\frac{d'}{dy}+z\frac{d'}{dz}\right)^iV_0'}{\lfloor i}=z^i-\frac{i(i-1)}{2}z^{i-2}(x^2+y^2)+\ldots,$$

in which it will be observed we have taken in the first case one form of the solid zonal harmonic, and in the second case the other. For this reason;—On looking at the second side of the general formula of § 4 we see that the operator on V differs from $\left(x\frac{d}{dx}+y\frac{d}{dy}+z\frac{d}{dz}\right)^i$ only in having $\frac{d'}{dx}\,\frac{d'}{dy}\,\frac{d'}{dz}$

in place of x, y, z. Now, if in the first of the above harmonics we replace x, y, z by these operators on V', we get

$$\left(\frac{d}{dx}\frac{d'}{dx}+\frac{d}{dy}\frac{d'}{dy}+\frac{d}{dz}\frac{d'}{dz}\right)^i V_0 V_0' = \frac{\lfloor 2i}{2^i \lfloor i}\left(\frac{d'}{dz}\right)^i V_0',$$

all the terms excepting the first disappearing by virtue of Laplace's equation, and now the advantage of expressing V' in terms of the second form of the zonal harmonic becomes apparent, for all the terms disappear by differentiation except the first, and the last written result becomes

$$\frac{\lfloor 2i}{2^i}.$$

Returning to the general formula of § 4, we finally obtain

$$\iint Q_i^2 dS = \frac{4\pi R^2}{2i+1}.$$

§ 6. Exactly similar reasoning applies in the case of the solid harmonics derived from the tesseral and sectorial forms, which are

$$\frac{\lfloor 2i}{2^{i+\sigma}\lfloor i \lfloor i}(\xi^\sigma+\eta^\sigma)\left\{z^{i-\sigma}-\frac{(i-\sigma)(i-\sigma-1)}{2(2i-1)}z^{i-\sigma-2}(x^2+y^2+z^2)+\ldots\right\},$$

$$\frac{\lfloor i+\sigma}{2^\sigma \lfloor i \lfloor \sigma}(\xi^\sigma+\eta^\sigma)\left\{z^{i-\sigma}-\frac{(i-\sigma)(i-\sigma-1)}{4(\sigma+1)}z^{i-\sigma-2}\xi\eta+\ldots\ldots\ldots\ldots\right\}.$$

These expansions are the values in different forms of $r^i Y_i^\sigma$, where

$$(-1)^i \lfloor i\, Y_i^\sigma = r^{i+1}\left(\frac{d}{dz}\right)^{i-\sigma}\left\{\left(\frac{d}{d\xi}\right)^\sigma+\left(\frac{d}{d\eta}\right)^\sigma\right\}\frac{1}{r}.$$

Since $\xi = x + jy$ and $\eta = x - jy$, we see that, according to the explanation of the last article, ξ, η will become respectively $\frac{d'}{dx}+j\frac{d'}{dy}$ and $\frac{d'}{dx}-j\frac{d'}{dy}$ opeating on V'; that is, $2\frac{d'}{d\eta}$, $2\frac{d'}{d\xi}$.

Hence $$\left(\frac{d}{dx}\frac{d'}{dx}+\frac{d}{dy}\frac{d'}{dy}+\frac{d}{dz}\frac{d'}{dz}\right)^i V_0 V_0'$$

becomes in this case

$$\frac{\lfloor 2i}{2^{i+\sigma}\lfloor i} 2^\sigma \left\{\left(\frac{d'}{d\xi}\right)^\sigma + \left(\frac{d'}{d\eta}\right)^\sigma\right\}\left(\frac{d'}{dz}\right)^{i-\sigma} V_0' = 2\frac{\lfloor 2i \lfloor i+\sigma \lfloor i-\sigma}{\lfloor i \lfloor i\, 2^{i+\sigma}}.$$

Hence $$\iint (Y_i^\sigma)^2 dS = \frac{8\pi R^2}{2i+1}\frac{\lfloor i+\sigma \lfloor i-\sigma}{2^\sigma \lfloor i \lfloor i}.$$

§ 7. The work in § 4 enables us to obtain easily the value of

$$\iint Q_i Y_i dS,$$

where Y_i is any surface harmonic, viz. it is

$$\frac{4\pi R^2}{2i+1}\frac{1}{\lfloor i}\left(\frac{d}{dz}\right)^i r^i Y_i,$$

which is obviously the same as

$$\frac{4\pi R^2}{2i+1}\frac{1}{\lfloor i}\frac{d^i}{dh_1 dh_2 \ldots dh_i} r^i Q_i,$$

that is $$\frac{4\pi R^2}{2i+1}(Y_i),$$

where (Y_i) is the value of Y_i at the pole of the Q harmonics.

§ 8. Since the determination of any harmonic depends upon successive differentiations of $\frac{1}{r}$, it is desirable that methods should be devised for obtaining those differentiations easily. Accordingly, in my former paper in the *Messenger* on this subject, I showed how those differentiations, instead of being made upon $\frac{1}{r}$ could be made upon $r^i Q_i$, where the pole of the zonal harmonic Q_i is the point on the sphere at which we want the general harmonic, the relation being in fact

$$(-1)^i \left[\frac{d}{dh}\right]^i \frac{1}{r} = \frac{1}{r^{2i+1}}\left[\frac{d}{dh}\right]^i Q_i r^i.$$

Now if (α, β) be the polar coordinates of the said pole, Q_i is a function of μ, the cosine of the distance of another point

(θ, ϕ) on the sphere measured from (α, β), that is, Q_i is a function of $\cos\alpha \cos\theta + \sin\alpha \sin\theta \cos(\phi - \beta)$.

The substitution of this value of μ in Q_i would be obviously a very laborious method of determining $Q_i r^i$ in a form suitable for the application of the theorem. We may, however, expand Q_i in terms of $2i + 1$ selected tesseral and sectorial harmonics, as is done by Thomson and Tait, or more simply by Ferrers (*Spherical Harmonics*, p. 88). This gives, in the notation here employed,

$$Q_i = (P_i)\, P_i + \ldots + \frac{2^{\sigma-1} \lfloor i \, \lfloor i}{\lfloor i+\sigma \, \lfloor i-\sigma} \{(Y_i^\sigma)\, Y_i^\sigma + (Z_i^\sigma)\, Z_i^\sigma\} + \ldots,$$

where $r^i P_i$ is now the second of the expressions in § 5, $r^i Y_i^\sigma$ is the second of those in § 6, and $r^i Z_i^\sigma$ is the same as $r^i Y_i^\sigma$ if we put $-\sqrt{(-1)}\,(\xi^\sigma - \eta^\sigma)$ instead of $\xi^\sigma + \eta^\sigma$ in the expansion of the latter. The brackets indicate that the values of α, β are to be put instead of θ, ϕ in the expressions P, Y, Z. It is obvious any differentiations upon $Q_i r^i$ can now be easily determined. In other words, we have any assigned harmonic expressed in terms of the selected tesseral and sectorial harmonics.

ON THE OCCURRENCE OF THE HIGHER TRANSCENDENTS IN CERTAIN MECHANICAL PROBLEMS.

By *W. H. L. Russell, F.R.S.*

(Continued from p. 21).

(4) A FRUSTRUM of a paraboloid of revolution rolls on a rough horizontal plane, determine the motion when the centre of gravity of the paraboloid coincides with the focus.

The motion is supposed throughout the investigations in this paper (with the exception of the first) to take place in parallel planes. Let then the line of intersection of the plane, in which (S) the focus of the paraboloid moves with the horizontal plane be taken for the axis of (x), A a point in it for the origin; let also P be the point of contact of paraboloid and horizontal plane, and B the vertex of the paraboloid. Let $AN = x$ and $NS' = y$ be the coordinates of S: $BSN = NSP = \theta$, the angle which the axis of the paraboloid makes with the vertical, also let $SB = a$, $SP = r$, then

the equations which determine the motion of the body are as follows:

$$\frac{d^2x}{dt^2} = -\frac{F}{M}, \quad \frac{d^2y}{dt^2} = -g + \frac{P}{M},$$

$$\frac{d^2\theta}{dt^2} = \frac{Fr\cos\theta}{Mk^2} - \frac{Pr\sin\theta}{Mk^2},$$

when F and P are the friction and pressure: the geometrical conditions are thus found.

Let us suppose for an instant the point P fixed while the axis of the paraboloid is rolling from an angle (θ) with the axis of (x) to an angle $\theta + d\theta$, then S describes a small space perpendicular to SP and equal to $rd\theta$, the resolved part of this in the direction of the axis of (x) is $rd\theta\cos\theta$; hence, the geometrical conditions are $\frac{dx}{d\theta} = r\cos\theta$, $y = \frac{a}{\cos\theta}$; combining these equations with the equations of motion we obtain the equation of *vis viva*, which proves the correctness of our reasoning. Substituting in the equation of *vis viva*, putting $u = \cos\theta$, we find

$$t = -\frac{1}{\sqrt{(2g)}}\int du \left\{\frac{a^2 + k^2u^4}{(1-u^2)\,u^3\,(cu-a)}\right\}^{\frac{1}{2}}.$$

(5) A sphere rolls down a cylinder whose base placed in a vertical position is a parabola, to determine the motion.

Let BP be a vertical section of the parabolic cylinder passing the centre of the sphere C, B the vertex of the parabola, BA the axis of the parabola supposed horizontal, A the focus, P the point of contact of sphere and cylinder, T the point of intersection of CP and BA. Take A for the origin, AB for the axis of (x), and let AN, NP be the coordinates of P. Also let $CTA = \theta$, $AB = a$, $CP = r$, then $AN = 2a - \frac{a}{\cos^2\theta}$, $NP = 2a\tan\theta$. Now let (x) and (y) be the coordinates of C, ϕ the angle through which any radius of the sphere rolls, then the equations of motion are

$$\frac{d^2x}{dt^2} = -\frac{F\sin\theta}{M} + \frac{R\cos\theta}{M},$$

$$\frac{d^2y}{dt^2} = \frac{F\cos\theta}{M} + \frac{R\sin\theta}{M} - g,$$

$$\frac{d^2\phi}{dt^2} = \frac{Fr}{Mk^2},$$

where F and R have their usual significations, also we have

$$-\frac{d\phi}{d\theta}=\frac{1}{r}\left\{r+\frac{2a}{\cos^3\theta}\right\}.$$

To explain this equation we remark that it is well known that if a sphere (radius ρ') roll upon a sphere (radius ρ), and if θ be the angle through which any radius ρ' rolls, while the line joining the centres of the two spheres describes an angle θ, then $\phi=\frac{\rho+\rho'}{\rho'}\theta$. In the present case instead of ρ we make use of the radius of curvature of the parabola at the point of contact, we take the angles infinitely small, and since θ is measured in a direction contrary to that of the rolling sphere we take $d\theta$ negative.

The other equations of condition are

$$x=r\cos\theta+2a-\frac{a}{\cos^2\theta},$$

$$y=r\sin\theta+2a\tan\theta.$$

In order that the equation of *vis viva* may hold, we must have

$$-\sin\theta\frac{dx}{d\theta}+\cos\theta\frac{dy}{d\theta}+r\frac{d\phi}{d\theta}=0,$$

$$\cos\theta\frac{dx}{d\theta}+\sin\theta\frac{dy}{d\theta}=0,$$

these equations are easily seen to be identically true, and we are therefore able to write down the equation for *vis viva*

$$\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}+k^2\frac{d\phi^2}{dt^2}=2g(c-y),$$

and we obtain finally

$$t=\frac{\sqrt{(r^2+k^2)}}{\sqrt{(2g)}}\int\frac{(r\cos^3\theta+2a)\,d\theta}{r\cos^3\theta\sqrt{(c-r\sin\theta-2a\tan\theta)}},$$

which may be reduced to an algebraical form by putting $u=\tan\frac{1}{2}\theta$.

(6) A heavy rod fastened to a hinge at one extremity presses on a semi-elliptic cylinder moving on a smooth horizontal plane containing the hinge, to determine the motion.

As the rod descends, we suppose the flat surface of the semi-elliptic cylinder, made by a plane passing through the

axis of the elliptic cylinder so as to cut off the greatest area, to glide along the horizontal plane. Let C be the centre of the ellipse, H the hinge, HP the beam pressing at P upon the semi-cylinder. Let $HC=x$, θ be the angle which the normal at P makes with the major axis, $2r$ the length of the beam to the plane, and the equations of motion are

$$\frac{d^2\theta}{dt^2}=\frac{gr\sin\theta}{k^2}-\frac{P}{mk^2}\cdot\frac{b\tan\theta}{a\sqrt{(1-e^2\sin^2\theta)}},$$

$$\frac{d^2x}{dt^2}=\frac{P}{m'}\cos\theta,$$

where

$$x=\frac{b^2}{a}\frac{\sin\theta\tan\theta}{\sqrt{(1-b^2\sin^2\theta)}}+\frac{a\cos\theta}{\sqrt{(1-b^2\sin^2\theta)}}=\frac{a\sqrt{(1-e^2\sin^2\theta)}}{\cos\theta},$$

from whence $$\frac{dx}{d\theta}=\frac{a(1-e^2\sin\theta)}{\cos^2\theta\sqrt{(1-e^2\sin^2\theta)}};$$

by aid of this equation we immediately deduce the equation of *vis viva* from the equations of motion, and we obtain finally

$$t=\frac{1}{\sqrt{(2mgr)}}\int\frac{d\theta}{\cos^2\theta}\left\{\frac{mk^2\cos^4\theta(1-e^2\sin^2\theta)+m'a^2(1-e^2)^2\sin^2\theta}{(1-e^2\sin^2\theta)(\cos a-\cos\theta)}\right\}^{\frac{1}{2}},$$

which may be reduced to an algebraical form by putting $\cos\theta=u$.

(To be continued.)

ON EULERIAN NUMBERS.

By *M. Edouard Lucas.*

1. If we put

$$\sec x=1+a_2x^2+a_4x^4+a_6x^6+\ldots\ \&c.,$$

we have, on multiplying the left-hand side of this equation by $\cos x$, and the right-hand side by the series for $\cos x$, the relation

$$a_{2n}-\frac{a_{2n-2}}{2\,!}+\frac{a_{2n-4}}{4\,!}-\ldots\pm\frac{a_2}{(2n-2)\,!}\mp\frac{1}{(2n)\,!}=0.$$

From this relation Mr. Glaisher has deduced an expression for a_{2n} as a determinant of the n^{th} order (*Messenger*, vol. VI., p. 52).

The Eulerian numbers are, in absolute value, given by the formula

$$E_{2n} = (-1)^n (2n)!\, a_{2n}.$$

We thus have, changing x into xi, the symbolic formula

$$\frac{2}{e^x + e^{-x}} = e^{Ex},$$

in the development of which the exponents of E are to be replaced by suffixes, and E_0 by unity. Getting rid of the denominators, we find, for n positive, the recurring relation

$$(E+1)^n + (E-1)^n = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

leading to the determinant

$$E_{2n} = (-1)^n \begin{vmatrix} 1, & 1, & 0, & 0, & 0, \ldots \\ 1, & 6, & 1, & 0, & 0, \ldots \\ 1, & 15, & 15, & 1, & 0, \ldots \\ 1, & 28, & 70, & 28, & 1, \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix} \quad (n \text{ rows})^*.$$

This determinant is formed of lines of even rank and of columns of uneven rank of the arithmetical triangle.

We have also the symbolic formula

$$2\{-1^n + 3^n - 5^n + 7^n + \ldots + (4x-1)^n\} = (4x+E)^n - E^n;$$

and, in addition, the formulæ

$$\frac{1}{1^{2n}} - \frac{1}{3^{2n}} + \frac{1}{5^{2n}} - \frac{1}{7^{2n}} + \ldots = \frac{(-1)^n \pi^{2n+1} E_{2n}}{2^{n+2}(2n)!},$$

$$\int_0^\infty \frac{x^{2n}\,dx}{e^{\pi x} + e^{-\pi x}} = \pm \frac{E_{2n}}{2^{2n+2}}.$$

2. Eulerian numbers are integers and they are uneven. Sherk has demonstrated that they end alternately in the figures 1 and 5. These properties can be proved as follows:

We deduce from the relation (1) for p prime the congruence

$$E_{p-1} + E_{p-3} + E_{p-5} + \ldots + E_2 + E_0 \equiv 0, \pmod{p};$$

whence, denoting by A_p the sum of the first p Eulerian numbers taken with their proper signs,

$$A_{p-1} \equiv 0, \pmod{p}.$$

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* This value of E_{2n} as a determinant was given by Mr. Hammond in his paper 'On the relation between Bernoulli's numbers and the Binomial coefficients,' *Proceedings of the London Mathematical Society*, vol. VII., p. 18, (1875).—ED.

The first values are given by the formulæ

$$E_2 + E_0 = 0,$$
$$E_4 + 6E_2 + E_0 = 0,$$
$$E_6 + 15E_4 + 15E_2 + E_0 = 0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

whence, starting from E_{p-1},

$$\left.\begin{aligned} E_{p+1} + E_0 &\equiv 0, \\ E_{p+3} + 3E_{p+1} + 3E_2 + E_0 &\equiv 0, \\ E_{p+5} + 10E_{p+3} + 5E_{p+1} + 5E_4 + 10E_2 + E_0 &\equiv 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \end{aligned}\right\} \text{(mod. } p\text{)}.$$

The comparison of these two systems of formulæ gives successively

$$\left.\begin{aligned} E_{p+1} &\equiv E_2, \\ E_{p+3} &\equiv E_4, \\ E_{p+5} &\equiv E_6, \\ \cdots\cdots & \end{aligned}\right\} \text{(mod. } p\text{)}.$$

We have, in general,

$$E_{2n} \equiv E_{2n+k(p-1)}, \text{ (mod. } p\text{)},$$

whatever value the positive integer k may have, and consequently:

Theorem. The residues of the Eulerian numbers, for any prime modulus whatever, reproduce themselves periodically in the same order, just as the residues of powers.

These considerations are applicable, in general, to the differential coefficients of a rational fraction of e^x, but under certain conditions, as in the case of

$$\frac{\phi(1)}{\phi(e^x)}.$$

When $\phi(1)$ is zero, as in the development of $\frac{1}{1-e^x}$, the theorem does not hold; the differential coefficients are no longer integers and contain in the denominators an indefinite series of prime numbers; it is so, for example, with the Bernoullian numbers.

Paris, *November*, 1877.

AN EXTENSION OF ARBOGAST'S METHOD OF DERIVATIONS.

By *J. J. Thomson*, Trinity College, Cambridge.

BY Arbogast's method if we have a series

$$v = a + bx + \frac{cx^2}{\lfloor 2} + \frac{dx^3}{\lfloor 3} + \&c.$$

We are able to calculate the successive coefficients in the expansion of any function of v in ascending powers of x. The rule is, each coefficient is derived from the preceding by differentiating it with respect to x and putting

$$\frac{da}{dx} = b, \frac{db}{dx} = c, \frac{dc}{dx} = d, \&c.$$

in the result. The object of the following paper is to demonstrate a similar rule when there are any number of variables in the series to be expanded.

Let $u = c_0 + c_x x + c_y y + \dfrac{c_{2x} x^2 + 2c_{x,y} xy + c_{2y} y^2}{\lfloor 2}$

$$+ \frac{c_{3x} x^3 + 3c_{2x,y} x^2 y + 3c_{x,2y} xy^2 + c_{3y} y^3 +}{\lfloor 3} \&c.$$

$$+ \frac{c_{nx} x^n + nc_{(n-1)x,y} x^{n-1} y + \dfrac{n(n-1)}{\lfloor 2} c_{(n-2)x,2y} x^{n-2} y^2}{\lfloor n} + \&c.,$$

it is required to find the coefficients in the expansion of $\phi(u)$ in ascending powers of x, y.

We notice that when x and y both $= 0$,

$$\left(\frac{d}{dx}\right)^m \left(\frac{d}{dy}\right)^n u = c_{mx,ny} \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Let $\qquad \phi(u) = F(x, y)$.

By Maclaurin's theorem we have

$$\phi(u) = F(0, 0) + x \left|\frac{dF}{dx}\right|_0 + y \left|\frac{dF}{dy}\right|_0 + \frac{1}{\lfloor 2} \left\{ x^2 \left|\frac{d^2F}{dx^2}\right|_0 \right.$$

$$\left. + 2xy \left|\frac{d^2F}{dxdy}\right|_0 + y^2 \left|\frac{d^2F}{dy^2}\right|_0 \right\} + \&c.,$$

the coefficient of

$$\frac{n(n-1)\ldots(n-m+1)}{\lfloor m} \frac{x^{n-m} y^m}{\lfloor n} \text{ is } \left|\frac{d^n F}{dx^{n-m} dy^m}\right|_0 = \left|\frac{d^n \phi(u)}{dx^{n-m} dy^m}\right|_{x=0,\, y=0}.$$

Let us denote $\dfrac{d^{r+s}}{dx^r dy^s}u$ when x and y are both zero by $u_{rx, sy}$; then by (1) $u_{rx, sy} = c_{rx, sy}$

if we assume
$$\left.\begin{aligned} \frac{d}{dx}(c_{rx, sy}) &= c_{(r+1)x, sy} \\ \frac{d}{dy}(c_{rx, sy}) &= c_{rx, (s+1)y} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$\left|\frac{d^n\phi(u)}{dx^{n-m}dy^m}\right|_{x=0,\, y=0}$$

will evidently
$$= \frac{d^n\phi(c_0)}{dx^{n-m}dy^m},$$

because
$$|u|_{x=0,\, y=0} = c_0 \text{ and } \left|\frac{d^{s+r}u}{dx^s dy^r}\right|_{x=0,\, y=0} = c_{sx, ry} = \frac{d^{s+r}c_0}{dx^s dy^r} \text{ by (2)};$$

but $\left|\dfrac{d^n\phi(u)}{dx^{n-m}dy^m}\right|_{x=0,\, y=0}$ is the coefficient of $\dfrac{n(n-1)\ldots(n-m+1)}{\underline{|m}}$ $\dfrac{x^{n-m}y^m}{\underline{|n}}$ in the expansion of $\phi(u)$; hence the coefficient of $\dfrac{n(n-m)\ldots(n-m+1)}{\underline{|m}}\dfrac{x^{n-m}y^m}{\underline{|n}}$ is got by differentiating $\phi(c_0)$ – m times with respect to x and m times with respect to y, remembering the conventions (2).

An example may make this clearer:—suppose the coefficient of $2xy$ in the expansion of sin u is required.

By the theorem it $= \dfrac{1}{\underline{|2}}\dfrac{d^2 \sin c_0}{dx\,dy}$,

$$\frac{d \sin c_0}{dy} = \cos c_0 \frac{dc_0}{dy} = \cos c_0 . c_y,$$

$$\frac{d}{dx}(\cos c_0 c_y) = -\sin c_0 \frac{dc_0}{dx} c_y + \cos c_0 \frac{dc_y}{dx}$$

$$= -\sin c_0 . c_x . c_y + \cos c_0 . c_{xy};$$

therefore the coefficient of $2xy$ in the expansion of sin u

$$= \frac{1}{\underline{|2}}\{\cos c_0 . c_{xy} - \sin c_0 . c_x c_y\}.$$

It is evident that a similar theorem will hold for any number of variables.

ON A FORMULA IN ELLIPTIC FUNCTIONS.

By *J. W. L. Glaisher.*

1. FROM the equation

$$\Theta\,(x+y)\,\Theta\,(x-y)=\frac{\Theta^2 x\,\Theta^2 y}{\Theta^2 0}\,(1-k^2\,\mathrm{sn}^2 x\,\mathrm{sn}^2 y),$$

we have

$$\Theta 2\,(u+v)\,\Theta 2\,(u-v)=\frac{\Theta^2 2u\,\Theta^2 2v}{\Theta^2 0}\,(1-k^2\,\mathrm{sn}^2 2u\,\mathrm{sn}^2 2v)\ldots(1),$$

$$\Theta 2\,(u+v)=\frac{\Theta^4\,(u+v)}{\Theta^8 0}\,\{1-k^2\,\mathrm{sn}^4\,(u+v)\},$$

$$\Theta 2\,(u-v)=\frac{\Theta^4\,(u-v)}{\Theta^3 0}\,\{1-k^2\,\mathrm{sn}^4\,(u-v)\},$$

$$\Theta 2u=\frac{\Theta^4 u}{\Theta^8 0}\,(1-k^2\,\mathrm{sn}^4 u),\quad \Theta\,(2v)=\frac{\Theta^4 v}{\Theta^8 0}\,(1-k^2\,\mathrm{sn}^2 v),$$

whence substituting for $\Theta 2\,(u+v)$, $\Theta 2\,(u-v)$, $\Theta 2u$, $\Theta 2v$ their values in (1), we have the formula

$$1-k^2\,\mathrm{sn}^2 2u\,\mathrm{sn}^2 2v$$

$$=\frac{\{1-k^2\,\mathrm{sn}^4\,(u+v)\}\{1-k^2\,\mathrm{sn}^4\,(u-v)\}}{(1-k^2\,\mathrm{sn}^4 u)^2\,(1-k^2\,\mathrm{sn}^4 v)^2}\,(1-k^2\,\mathrm{sn}^2 u\,\mathrm{sn}^2 v)^4\ldots(2).$$

This equation may also be obtained by a double application of the formula

$$1-k^2\,\mathrm{sn}^2\,(u+v)\,\mathrm{sn}^2\,(u-v)=\frac{(1-k^2\,\mathrm{sn}^4 u)\,(1-k^2\,\mathrm{sn}^4 v)}{(1-k^2\,\mathrm{sn}^2 u\,\mathrm{sn}^2 v)^2},$$

which is readily proved.

2. Writing

$$f(u,\,v)=1-k^2\,\mathrm{sn}^2 u\,\mathrm{sn}^2 v,\quad \phi u=1-k^2\,\mathrm{sn}^4 u,$$

we see from (2) that

$$f\,(4u,\,4v)=\frac{\phi 2\,(u+v)\,\phi 2\,(u-v)}{(\phi 2u\,\phi 2v)^2}\,f^4\,(2u,\,2v)$$

$$=\frac{\phi 2\,(u+v)\,\phi^4\,(u+v)\,.\,\phi 2\,(u-v)\,\phi^4\,(u-v)}{(\phi 2u\,\phi^4 u\,.\,\phi 2v\,\phi^4 v)}\,f^{16}\,(u,\,v),$$

$$f(8u,\,8v)=\&\text{c.}=\{\quad\}\,f^{64}\,(u,\,v),$$

....

Thus, starting with $f(u, v)$, and expressing it successively in terms of $f(\frac{1}{2}u, \frac{1}{2}v), f(\frac{1}{4}u, \frac{1}{4}v), \ldots$, and observing that when n is infinite,

$$f^{4^n}\left(\frac{u}{2^n}, \frac{v}{2^n}\right) = \left(1 - k^2 \operatorname{sn}^2 \frac{u}{2^n} \operatorname{sn}^2 \frac{v}{2^n}\right)^{4^n}$$

$$= \left(1 - \frac{k^2u^2v^2}{4^{2n}}\right)^{4^n} = 1,$$

we obtain the theorem, that if the infinite product

$$(1 - k^2 \operatorname{sn}^4 u)(1 - k^2 \operatorname{sn}^4 \tfrac{1}{2}u)^4 (1 - k^2 \operatorname{sn}^4 \tfrac{1}{4}u)^{16} (1 - k^2 \operatorname{sn}^4 \tfrac{1}{8}u)^{64} \ldots$$

be denoted by $\chi(u)$, then

$$1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v = \frac{\chi\frac{1}{2}(u+v)\,\chi\frac{1}{2}(u-v)}{\chi^2\frac{1}{2}u\,\chi^2\frac{1}{2}v}.$$

SUB-FACTORIAL *N*.

By *W. Allen Whitworth.*

1. A NEW symbol in algebra is only half a benefit unless it have a new name. We believe that the symbol $\lfloor n$ as an abbreviation of the continued product of the first n integers, was long in use before the name *factorial n* was adopted. But until it received its name it appealed only to the eye and not to the ear, and in reading aloud could only be described by a periphrasis.

In introducing the symbol $\|\underline{n}$, we propose at once to call it *sub-factorial n.*

2. The close alliance of $\lfloor n$ and $\|\underline{n}$ will be immediately seen from the following, which may be taken as the definition of the sub-factorials. It will be observed, that the omission of the words in italics will give us the factorials.

Write down 1, *and subtract* 1; the result is $\|\underline{1}$.

Multiply by 2, *and add* 1; the result is $\|\underline{2}$.

Multiply by 3, *and subtract* 1; the result is $\|\underline{3}$.

Multiply by 4, *and add* 1; the result is $\|\underline{4}$.

Multiply by 5, *and subtract* 1; the result is $\|\underline{5}$.

Multiply by 6, *and add* 1; the result is $\|\underline{6}$.

And so on.

And generally $\underline{\|n}$ may be obtained from $\underline{\|n-1}$ by multiplying by n and adding $(-1)^n$.

3. We have
$$\underline{\|n} = n\,\underline{\|n-1} \pm 1,$$
and
$$\underline{\|n-1} = (n-1)\,\underline{\|n-2} \mp 1.$$
Therefore, by addition,
$$\underline{\|n} + \underline{\|n-1} = \underline{\|n-1} + (n-1)\,\underline{\|n-2},$$
or
$$\underline{\|n} = (n-1)\left(\underline{\|n-1} + \underline{\|n-2}\right),$$
which shews that $\underline{\|n}$ *is always divisible by* $n-1$.

4. Again, we have
$$\underline{\|n} = n\,\underline{\|n-1} + (-1)^n.$$
Divide by $\lfloor\underline{n}$, and we get
$$\frac{\underline{\|n}}{\lfloor\underline{n}} = \frac{\underline{\|n-1}}{\lfloor\underline{n-1}} + \frac{(-1)^n}{\lfloor\underline{n}},$$
Similarly
$$\frac{\underline{\|n-1}}{\lfloor\underline{n-1}} = \frac{\underline{\|n-2}}{\lfloor\underline{n-2}} + \frac{(-1)^{n-1}}{\lfloor\underline{n-1}},$$
$$\frac{\underline{\|n-2}}{\lfloor\underline{n-2}} = \frac{\underline{\|n-3}}{\lfloor\underline{n-3}} + \frac{(-1)^{n-2}}{\lfloor\underline{n-2}},$$
and so on,

Finally,
$$\frac{\underline{\|2}}{\lfloor\underline{2}} = \frac{\underline{\|1}}{\lfloor\underline{1}} + \frac{1}{\lfloor\underline{2}},$$
and
$$\frac{\underline{\|1}}{\lfloor\underline{1}} = 1 - \frac{1}{\lfloor\underline{1}}.$$
Therefore, by addition,
$$\frac{\underline{\|n}}{\lfloor\underline{n}} = 1 - \frac{1}{\lfloor\underline{1}} + \frac{1}{\lfloor\underline{2}} - \frac{1}{\lfloor\underline{3}} + \frac{1}{\lfloor\underline{4}} - \&c.\ldots + \frac{(-1)^n}{\lfloor\underline{n}}.$$

5. If e be the base of Napierian logarithms we have by the exponential theorem
$$\frac{1}{e} = 1 - \frac{1}{\lfloor\underline{1}} + \frac{1}{\lfloor\underline{2}} - \frac{1}{\lfloor\underline{3}} + \&c.\ldots\text{to infinity}.$$

Therefore, if n be even, $\dfrac{\underline{\|n}}{\underline{|n}} > \dfrac{1}{e}$;

if n be odd $\dfrac{\underline{\|n}}{\underline{|n}} < \dfrac{1}{e}$;

and if n be increased indefinitely

$$\text{limit}_{n=\infty} \frac{\underline{\|n}}{\underline{|n}} = \frac{1}{e}.$$

6. The rapidity with which the ratio of the factorial to the sub-factorial converges to equality with e is seen by a glance at the following table, in which the values of the first twelve factorials and sub-factorials are registered, together with the quotients $\underline{|n} \div \underline{\|n}$ and $\underline{\|n} \div \underline{|n}$ to seven places of decimals.

n	$\underline{\lvert n} \div \underline{\lVert n}$	$\underline{\lvert n}$	$\underline{\lVert n}$	$\underline{\lVert n} \div \underline{\lvert n}$	n
1	∞	1	0	0	1
2	2·0000000	2	1	0·5000000	2
3	3·0000000	6	2	0·3333333	3
4	2·6666666	24	9	0·3750000	4
5	2·7272727	120	44	0·3666666	5
6	2·7169811	720	265	0·3680555	6
7	2·7184466	5040	1854	0·3678571	7
8	2·7182623	40320	14833	0·3678819	8
9	2·7182836	362880	133496	0·3678791	9
10	2·7182816	3628800	1334961	0·3678794	10
11	2·7182818	39916800	14684570	0·3678794	11
12	2·7182818	479001600	176214841	0·3678794	12
...	...	...	...	...	...
∞	2·7182818	∞	∞	0·3678794	∞

7. Sub-factorials chiefly occur in connexion with permutations.

For example, if n terms are to be arranged in order, this can be done in $\underline{|n}$ ways. If they are again to be arranged in order *so that no term shall be where it was before*, this can be done in $\underline{\|n}$ ways (*Choice and Chance*, Prop. XXXII). If they are to be arranged so that no term may be followed by the term which originally followed it, this can be done in $\underline{\|n} + \underline{\|n-1}$ ways (*Choice and Chance*, Prop. XXXIII).

ON THE PORISM OF THE RING OF CIRCLES TOUCHING TWO CIRCLES.

By *H. M. Taylor, M.A.*

It is easily seen that the radius of a circle which touches two concentric circles of radii α and β is $\frac{1}{2}(\alpha-\beta)$, and that the distance of its centre from the common centre of the two circles is $\frac{1}{2}(\alpha+\beta)$. It follows therefore that if $\frac{\alpha-\beta}{\alpha+\beta}=\sin\frac{\pi}{n}$, a complete ring of tangent circles could be described between the two circles, each tangent circle touching those on either side of it (fig. 9).

Now by inversion we can deduce the condition that such a ring of tangent circles can be described in the case of a pair of non-concentric circles; and it follows, at once, that if one such ring is possible, an infinite number of such rings are possible, or, in fact, that we can begin a ring with any tangent circle.

Now let us assume that two concentric circles of radii α, β satisfying the relation $\alpha-\beta=(\alpha+\beta)\sin\frac{\pi}{n}$, be inverted with respect to a pole O distant d from the common centre; then if h be the constant of inversion, and the radii of the inverse circles be a and b, and c be the distance between their centres; and if the line through O through the centres of these circles cut them in D, C, B, A (figs. 10 and 11); then

$$OA=\frac{h^2}{d-\alpha}=x+2a \quad\text{......................(1)},$$

$$OB=\frac{h^2}{d-\beta}=x+a-c+b \quad\text{............(2)},$$

$$OC=\frac{h^2}{d+\beta}=x+a-c-b \quad\text{............(3)},$$

$$OD=\frac{h^2}{d+\alpha}=x \quad\text{.........................(4)};$$

also
$$\frac{\alpha-\beta}{\alpha+\beta}=\sin\frac{\pi}{n},$$

or
$$\frac{\alpha}{\beta}=\frac{1+\sin\frac{\pi}{n}}{1-\sin\frac{\pi}{n}}=m \text{ suppose} \quad\text{...............(5)}.$$

To find the required relation we must eliminate from these equations the 5 quantities h, d, α, β, x, which are only equivalent to 4 quantities.

In fact we may put $h=1$ and then eliminate the remaining 4 quantities.

From (1) and (4)

$$\frac{\alpha}{d^2-\alpha^2}=a \quad\text{..............................(6)},$$

$$\frac{d}{d^2-\alpha^2}=x+a \quad\text{........................(7)};$$

from (2) and (3)

$$\frac{\beta}{d^2-\beta^2}=b \quad\text{..............................(8)},$$

$$\frac{d}{d^2-\beta^2}=x+a-c \quad\text{....................(9)};$$

from (7) and (9)

$$\frac{d}{d^2-\alpha^2}-\frac{d}{d^2-\beta^2}=c.$$

Therefore by substituting from (6) and (8)

$$\frac{a}{\alpha}-\frac{b}{\beta}=\frac{c}{d},$$

or

$$c=\frac{d}{\alpha}(a-mb) \quad\text{.......................(10)},$$

similarly

$$c=\frac{d}{\beta}\left(\frac{a}{m}-b\right) \quad\text{.....................(11)}.$$

Therefore from (6), (8), (10), and (11)

$$\frac{ab}{\alpha\beta}=\frac{d^2-\alpha^2}{d^2-\beta^2}=\frac{1-\left(\frac{\alpha}{d}\right)^2}{1-\left(\frac{\beta}{d}\right)^2}=\frac{c^2-(a-mb)^2}{c^2-\left(\frac{a}{m}-b\right)^2},$$

or

$$\frac{mb}{a}=m^2\frac{c^2-(a-mb)^2}{c^2m^2-(a-mb)^2},$$

$$mc^2(a-bm)=(am-b)(a-bm)^2,$$

or, since a is not equal to bm,

$$mc^2=(a-bm)(am-b) \quad\text{..............(12)}.$$

The only remark which it is necessary to make to

complete the proof that this relation between the radii and the distance between the centres of two circles is sufficient for the existence of a ring of tangent circles is that any two non-intersecting circles can always be inverted into two concentric circles, if we take either of two particular points for the pole of inversion. The distances of these points from the centre of the circle whose radius is b are the roots of the equation

$$\frac{a^2}{c+y} - \frac{b^2}{y} = c,$$

which are always real, if the given circles do not intersect.

In fig. 9, $n=6$ and $\alpha=3\beta$. Figs. 10 and 11 are obtained by inverting fig. 9 with respect to a pole, where $d=5\beta$; whence it follows that $AB : BC : CD = 6 : 2 : 1$.

It may be added that, if we write $m\frac{\pi}{n}$ for $\frac{\pi}{n}$ in the above equations, we obtain the condition for the existence of a ring of n circles encircling the inner of the two given ones m times.

A THEOREM IN AREAS INCLUDING HOLDITCH'S, WITH ITS ANALOGUE IN THREE DIMENSIONS.

By *E. B. Elliott, M.A.* Queen's College, Oxford.

I. HOLDITCH'S theorem in the extended form given in Williamson's Calculus establishes a relation connecting the areas of the closed curves traced by three given points in a rod of constant length as that rod moves in one plane through any cycle of positions back to its original one.* It is easy, as follows, to connect the areas passed round by three tracing points in a varying straight line in the more general case, when, instead of being at fixed distances, they are only at distances whose ratios remain constant, and which return to their initial lengths and positions after a complete cycle. In other words, we may replace the rod by a uniform elastic string.

Let (x_1y_1) (x_2y_2) be the coordinates referred to rectangular axes of two points A, B in their plane, and let (xy) be those

* The limitation that in this motion it must have rotated through an angle 2π is unnecessary.

of C which divides AB in the constant ratio $m:n$ (it may, of course, be either internal or external to AB); and give the positions a slight change in the plane of reference. Then

$$\begin{aligned}(m+n)^2\,ydx\\ &= (my_2+ny_1)(mdx_2+ndx_1)\\ &= m^2y_2dx_2+n^2y_1dx_1+mn(y_2dx_1+y_1dx_2)\\ &= m(m+n)\,y_2dx_2+n(m+n)\,y_1dx_1-mn(y_2-y_1)\,d(x_2-x_1)\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ldots\ldots\ldots\ldots\ldots\ldots(1).\end{aligned}$$

Now suppose A to travel all round the perimeter of a closed area (A), and B simultaneously all round that of another closed area (B), the two motions being quite independent and subject to no restrictions whatever, except that both be continuous, having no abrupt passage from one position to another finitely differing from it. C will then also travel simultaneously and continuously all round the perimeter of another closed area, which call (C). Integrating over a complete circuit we have then

$$\int y_1dx_1=(A),\ \int y_2dx_2=(B),\ \int ydx=(C).$$

Also, $\int(y_2-y_1)\,d(x_2-x_1)$ equals the area swept out by AB relatively to A, that is to say, the area enclosed by the path of a point always situated with regard to a fixed point, just as B is with regard to A. Call this relative area S; then it follows from (1) that

$$(m+n)^2(C)=m(m+n)(B)+n(m+n)(A)-mnS,$$

i.e.
$$(C)=\frac{m(B)+n(A)}{m+n}-\frac{mn}{(m+n)^2}S\ \ldots\ldots\ldots(2),$$

a result which may be stated thus. *Through any fixed point in the plane of a closed area S let radii vectores be drawn to all points of its perimeter; and let chords AB, parallel and equal to these radii vectores, be placed with one extremity A in each case in the perimeter of a closed area (A), and the other B on that of another (B).* The perimeter must be such that the points A, B so placed pass all round them respectively, and do not in either case return to their first positions from the same side as that towards which they left them. If in either case, that of B say, this is done, the area (B) must be replaced by zero. *Then (C) being the area enclosed by the trace of a point, always dividing AB in the constant ratio $m:n$, (A), (B), (C) are connected by the formula* (2).

Areas described in opposite senses of rotation must of course be taken as of opposite signs.

This relation may be expressed symmetrically in terms of any position, by writing $m : n : m+n = AC : CB : AB$.

Doing this it becomes

$$(C) = \frac{AC(B) + CB(A)}{AB} - AC.CB.\frac{S}{AB^2},$$

which, paying attention to sign, may be written

$$BC(A) + CA(B) + AB(C) = -BC.CA.AB\frac{S}{AB^2} \text{......(3)},$$

which is entirely symmetrical, $\frac{S}{AB^2}$ being the symbol of no linear dimensions, which operating on AB^2, BC^2 or CA^2 produces in each case the area swept out by the corresponding segment relatively to its extremity.

In the special case where AB is of constant length, and C divides it into two constant parts a, b so that S being a circular area described about its centre is $\pi(a+b)^2$; (2) and (3) become the known equivalent relations,

$$\left.\begin{aligned} (C) &= \frac{b(A) + a(B)}{a+b} - \pi ab \\ BC(A) + CA(B) + AB(C) &+ \pi BC.CA.AB = e \end{aligned}\right\} \text{.......(4).}$$

Numerous examples are easily deduced from (2) and (3), as for instance: *If C, C' divide the one internally and the other externally in the same constant ratio $m : n$ a line AB whose extremities move simultaneously and independently all round the perimeters of two closed areas (A) (B), the areas enclosed by their traces satisfy the relation*

$$(m+n)^2(C) + (m-n)^2(C') = 2\{m^2(B) + n^2(A)\}.$$

II. Proceeding now to three dimensions, suppose two closed surfaces of volumes (A) (B) respectively. Suppose also a series of points A to cover the whole surface of (A), and a series of corresponding points B to cover the whole surface of (B), in such a way that to positions of A at infinitely small distances all round any specified position correspond positions of (B) at infinitely small distances all round the corresponding position, the two systems subject to this one restriction having any independent laws of distribu-

tion over their respective surfaces whatever.* It is required to express as simply as possible the volume enclosed by the locus of a point dividing AB in any constant ratio $m : n$.

Taking rectangular axes, call A $(x_1y_1z_1)$, B $(x_2y_2z_2)$ and C the point dividing AB in the ratio $m : n$ (xyz). Then

$$(m+n)^3 zdxdy = (mz_2 + nz_1)(mdx_2 + ndx_1)(mdy_2 + ndy_1).$$

The right-hand side of this containing, when expanded, four distinct sets of terms, cannot be expressed more simply than as a sum of four volume elements, whereas in the analogous plane formula (1) three area elements were sufficient. The equation is then best written

$$\begin{aligned}(m+n)^3 zdxdy &= m(m^2-n^2)\,z_2dx_2dy_2 - n(m^2-n^2)\,z_1dx_1dy_1 \\ &\quad - \frac{mn}{2}(m-n)(z_2-z_1)(dx_2-dx_1)(dy_2-dy_1) \\ &\quad + \frac{mn}{2}(m+n)(z_2+z_1)(dx_2+dx_1)(dy_2+dy_1)\text{.......}(5).\end{aligned}$$

Now $zdxdy$ is the volume element of (C) the space bounded by the locus of C, $z_1dx_1dy_1$ the volume element of (A) and $z_2dx_2dy_2$ that of (B). Also $(z_2-z_1)\,d(x_2-x_1)\,d(y_2-y_1)$ is the volume element of the locus of B relative to A, that is to say, of the space which would be included by the locus of a point always situated with regard to a fixed point precisely as B is with regard to A; and $\frac{1}{8}(z_2+z_1)\,d(x_2+x_1)\,d(y_2+y_1)$ is the volume element of the locus of the middle point of AB. Writing then (M) for the volume enclosed by the locus of the middle point and V for the relative volume, it results from the integration of (5) over a complete series of positions that

$$\begin{aligned}(m+n)^3(C) &= m(m^2-n^2)(B) - n(m^2-n^2)(A) \\ &\quad - \tfrac{1}{2}mn(m-n)V + 4mn(m+n)(M),\end{aligned}$$

$$\begin{aligned}\text{or } (C) &= \frac{m-n}{(m+n)^2}\{m(B)-n(A)\} \\ &\quad + \frac{4mn}{(m+n)^2}(M) - \frac{mn(m-n)}{2(m+n)^3}V\text{...}(6).\end{aligned}$$

In the special case where AB is of constant length $a+b$ and C divides it into two given parts a, b, this becomes, since

* See however the concluding paragraph below.

V, being now a spherical volume described about its centre, equals $\frac{4}{3}\pi(a+b)^3$,

$$(C)=\frac{a-b}{(a+b)^2}\{a(B)-b(A)\}+\frac{4ab}{(a+b)^2}(M)-\frac{2\pi}{3}ab(a-b)\ldots(7).$$

The relation (6) involving as it does (M) as well as (A), (B) and (C) is one connecting the volumes corresponding to four points on a varying straight line, whereas the analogous relation in a plane (2) considered only three points. It is of course easy by eliminating of (M) from (6) and the similar relation connecting (A), (B), (M) and (D) the volume surrounded by the locus of a fourth point D, which divides AB in another constant ratio $m':n'$, to obtain the general relation as to four points on the line, neither of which is of necessity the middle point of the segment limited by any two others. This may be conveniently found in a symmetrical form in terms of any position, as follows.

In any position we have $AC:CB:AB=m:n:m+n$ so that (6) may be written

$$(C)=\frac{(AC-CB)AC}{AB^2}(B)-\frac{(AC-CB)CB}{AB^2}(A)$$
$$+\frac{4.AC.CB}{AB^2}(M)-\frac{AC.CB(AC-CB)}{2.AB^3}V,$$

$$i.e.\ \frac{(C)}{AC.CB}=\left(\frac{AC}{CB}-1\right)\frac{(B)}{AB^2}+\left(\frac{CB}{AC}-1\right)\frac{(A)}{AB^2}$$
$$+4\frac{(M)}{AB^2}-\tfrac{1}{2}(AC-CB)\frac{V}{AB^3}.$$

In like manner

$$\frac{(D)}{AD.DB}=\left(\frac{AD}{DB}-1\right)\frac{(B)}{AB^2}+\left(\frac{DB}{AD}-1\right)\frac{(A)}{AB^2}$$
$$+4\frac{(M)}{AB^2}-\tfrac{1}{2}(AD-DB)\frac{V}{AB^3}.$$

Thus, subtracting, we see that A, B, C, D being any four points on a varying straight line at distances in constant ratios to each other throughout the motion, which return after a complete cycle to their initial positions

$$\frac{(C)}{AC.CB}-\frac{(D)}{AD.DB}=\frac{AC.DB-AD.CB}{CB.DB}\cdot\frac{(B)}{AB^2}$$
$$+\frac{AD.CB-AC.DB}{AC.AD}\cdot\frac{(A)}{AB^2}+CD\cdot\frac{V}{AB^3},$$

or since

$$AC.DB - AD.CB = (AD - CD)(CB - CD) - AD.CB$$
$$= CD(CD - AD - CB)$$
$$= -AB.CD,$$

$$\frac{(C)}{AC.CB.CD} - \frac{(D)}{AD.DB.CD} = -\frac{(B)}{CB.DB.AB}$$
$$+ \frac{(A)}{AC.AD.AB} + \frac{V}{AB^3},$$

which, paying due regard to the signs of the segments, may be written

$$\frac{(A)}{AB.AC.AD} + \frac{(B)}{BA.BC.BD} + \frac{(C)}{CA.CD.CB}$$
$$+ \frac{(D)}{DA.DB.DC} = -\frac{V}{AB^3} \text{.........} (8),$$

a relation that is entirely symmetrical, for if $\frac{V}{AB^3}$ be denoted by v, this is of no dimensions in length, so that $v.AC^3$, $v.AD^3$, $v.BC^3$, &c., represent the volumes swept out relatively to their first written extremity in each case by AC, AD, BC, &c., just as $v.AB^3$ denotes that swept out by AB relatively to A.

In the special case where $ABCD$ is a rod of constant length, so that AB, AC, AD, BC, &c., are all constant, and consequently V is the spherical volume $\frac{4}{3}\pi AB^3$, this becomes

$$\frac{(A)}{AB.AC.AD} + \frac{(B)}{BA.BC.BD} + \frac{(C)}{CA.CB.CD}$$
$$+ \frac{(D)}{DA.DB.DC} + \frac{4}{3}\pi = 0 \text{......} (9),$$

the analogue of the second form of the special result (4) just as (7) is of the first.

In practically applying these formulæ to examples, caution is necessary in order to attach the right sign to each of the volumes involved. It will have been noticed that, throughout the above, the volume included by the locus of one point P relatively to another Q has been considered the negative of that included by the locus of Q relatively to P. From considerations of this kind, it will be clear that in formulæ (6) to (9) volumes generated by points of a shifting straight line on

one side of the instantaneous centre of rotation in any of the possible infinitesimal motions (I, for convenience, use language strictly applicable only to the special case of the kinematics of a rod of fixed length) being considered positive, those whose generating points are on the other side must be taken as negative.

It thus further appears, that for the formulæ to hold without modification the four generating points A, B, C, M or A, B, C, D, as the case may be, must be so restricted in position on their straight line, that in every shifting of this considered, the instantaneous centre lies between the same two of them: a restriction which has none analogous to it in the corresponding theorem of plane description first considered.

MATHEMATICAL NOTES.

An Elliptic Function Identity.

The following equation is true, identically,

$$\operatorname{sn}(\alpha-\beta)\operatorname{sn}\alpha\operatorname{sn}\beta+\operatorname{sn}(\beta-\gamma)\operatorname{sn}\beta\operatorname{sn}\gamma+\operatorname{sn}(\gamma-\alpha)\operatorname{sn}\gamma\operatorname{sn}\alpha$$
$$+\operatorname{sn}(\alpha-\beta)\operatorname{sn}(\beta-\gamma)\operatorname{sn}(\gamma-\alpha)=0.$$

M. M. U. WILKINSON.

Note on the Calculus of Functions.

I do not know whether attention has been drawn to the somewhat obvious fact that in solving functional equations we may replace the arbitrary constants by symbols of operation. The operations must not be algebraical, such as $\sqrt{(u)}$, $\sqrt{(1-u^2)}$, for these are taken into account in solving the equation; but they may be differentiations (positive or negative) which do not change when the subject is changed. For instance, take

$$\phi(x+y)=\phi(x)+\phi(y),$$

a solution of which is

$$\phi(x)=Cx.$$

This, however, is not a general solution, for ϕ includes all operative symbols which satisfy the distributive law, and in

the solution C may be replaced by any integral function of such symbols. Thus

$$\phi(x) = \Sigma_{ij} A_{ij} \frac{d^{i+j}x}{du^i dv^j},$$

A being constant, i, j integers, is a solution.

Again, a solution of

$$\phi(xy) = \phi(x) + \phi(y),$$

is

$$\phi(x) = C \log x.$$

Here, again, C may be replaced by any integral function of $\frac{d}{du}$, $\frac{d}{dv}$, &c.... For instance

$$\phi(x) = c \frac{d}{du} \log x = \frac{c}{x} \cdot \frac{dx}{du}$$

is a solution, since

$$\frac{c}{xy} \cdot \frac{d(xy)}{du} = \frac{c}{x} \cdot \frac{dx}{du} + \frac{c}{y} \cdot \frac{dy}{du}.$$

But C may not involve $\frac{d}{dx}$, an operator which changes with the subject. There is not a solution, for example, of the form

$$\phi(x) = c \frac{d}{dx} \log x = \frac{c}{x};$$

for the equation

$$\frac{c}{xy} = \frac{c}{x} + \frac{c}{y}$$

is not identically true.

H. W. LLOYD TANNER.

January, 1878.

Arithmetical Note.

The continued product

$$1.2...n-1.n, \ = \Pi(n),$$

cannot be a power of any integer, n being greater than 1. For if p be a prime $\Pi(p)$ cannot be a power since it contains a prime factor p, once and only once. For the same reason $\Pi(p+q)$ cannot be a power, if q be less than p. Nor can

even $\Pi(2p)$ be a power (square) if between p and $2p$ there is a prime, say p', for then $\Pi(2p)$ would contain the factor p' once and only once. Hence a necessary condition that $\Pi(n)$ should be a power for some value of n, is that some prime number p should be followed by a sequence of p composite numbers. This is obviously impossible when p is large; for small values of p, the following sequence of primes will shew the impossibility, since in it each prime (p) is followed by another less than $2p$,

$$2, 3, 5, 7, 13, 23, 43, 83, 163, \ldots .$$

H. W. Lloyd Tanner.

January, 1878.

Note on Arbogast's Method of Derivations.

It is an injustice to Arbogast to speak of his *first* method, as Arbogast's method;* there is really nothing in this, it is the straightforward process of expanding

$$\phi\left(a + bx + \frac{1}{1.2}cx^2 + \ldots\right)$$

by the differentiation of ϕu, writing $a, b, c, d \ldots$ in place of $u, \dfrac{du}{dx}, \dfrac{d^2u}{dx^2}, \dfrac{d^3u}{dx^3}$, &c. or say in place of u, u', u'', u''', &c. respectively; thus

$$\phi a,\ \phi' a.b,\ \tfrac{1}{2}\{\phi' a.c + \phi'' a.b^2\},\quad \tfrac{1}{6}\left\{\begin{matrix}\phi' a.d + \phi'' a.\ bc \\ \qquad + \phi'' a.2bc + \phi''' a.b^3\end{matrix}\right\}$$
$$= \tfrac{1}{6}\{\phi' a.d + \phi'' a.3bc + \phi''' a.b^3\}, \text{\&c.},$$

and in subsequent terms the number of additions necessary for obtaining the numerical coefficients increases with great rapidity.

That which is specifically Arbogast's method, is his *second* method, viz. here the coefficients of the successive powers of x in the expansion of $\phi(a + bx + cx^2 + dx^3 + \ldots)$, are obtained by the rule of the last and the last but one; thus we have

$$\phi a,\ \phi' a.b,\ \phi' a.c + \phi'' a.\tfrac{1}{2}b^2,\ \phi' a.d + \phi'' a.bc + \phi''' a.\tfrac{1}{6}b^3, \text{\&c.},$$

where each numerical coefficient is found directly, without an addition in any case.

A. Cayley.

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* See pp. 142, 143.

TRANSACTIONS OF SOCIETIES.

London Mathematical Society.

Thursday, December 13th, 1877.—C. W. Merrifield, Esq., F.R.S., *V.-P.*, in the chair. The Rev. W. Ellis, B.A., was elected a Member, and Mr. F. B. W. Phillips, B.A., Balliol College, Oxford, was proposed for election. Mr. S. Roberts, M.A., read a paper on Normals, which contained theorems depending on the invariants and covariants of the quartic equation representing a pencil of four normals to a conic, and drew attention to the remarkable cubic locus of the points of possible concurrence of three normals at the vertices of a given inscribed triangle. Dr. Hirst, F.R.S., and Mr. J. J. Walker spoke on the subject of the communication.—Prof. Cayley, F.R.S., read a paper on the Geometrical Representation of Imaginary Variables by a Real Correspondence of Two Planes. The communication was related to a former paper by the same writer, entitled "A Geometrical Illustration of a Theorem relating to an Irrational Function of an Imaginary Variable" (*Proceedings of London Mathematical Society*, t. VIII., pp. 212–214). Prof. Cayley was under the impression that the theory was a known one, but he has not found it anywhere set out in detail; he remarked that it was noticeable that, although intimately connected with, it is quite distinct from (and seems to go beyond) that of a Riemann's Surface.—A set of four models, presented by Dr. Zeuthen, was exhibited. Their title is "Quatre Modèles représentant des surfaces développables avec des renseignements sur la constructions des modèles et sur les singularités qu'ils représentent par V. Malthe Bruun and C. Crone avec quelques remarques sur les surfaces devéloppables et sur l'utilité des modèles par M. le Dr. Zeuthen (Copenhague)." Their construction is founded upon the description of a model given in Dr. Salmon's *Geometry of Three Dimensions* (p. 289, 3rd ed.), the invention of which is ascribed by Prof. Cayley to Prof. Blackburn. These models exhibit many of the chief singularities of developable surfaces.

Thursday, January 10th, 1878.—Lord Rayleigh, F.R.S., *President*, in the Chair. Mr. F. B. W. Phillips was elected a Member and Mr. R. R. Webb was admitted into the Society. The following papers were read: Mr. J. Hammond "On the meaning of the Differential Symbol D^n, when n is Fractional." (Prof. Cayley gave a few references to papers on the subject by Riemann, Schroeter, and others, and expressed his opinion that the matter had not yet been satisfactorily settled). Prof. Lloyd Tanner "On Partial Differential Equations with several Dependent Variables." Lord Rayleigh "On the Relation between the Functions of Laplace and Bessel, in § 783 of Thomson and Tait's *Natural Philosophy*, a suggestion is made to examine the transition from formulæ dealing with Laplace's spherical functions to the corresponding formulæ proper to a plane. It is evident at once, from this point of view, that Bessel's functions are merely particular cases of Laplace's more general functions, but the fact seems to be very little known. Mr. Ferrers in his elementary treatise on Spherical Harmonics, makes no mention of Bessel's functions, and Mr. Todhunter in his work on these functions states expressly that Bessel's functions are not connected with the main subject of the book. The object of the present paper was to point out briefly the correspondence of some of the formulæ. The Author showed that the Bessel's function of zero order, (J_0), is the limiting form of Legendre's function $P_n(\mu)$ when n is indefinitely great and μ ($= \cos\theta$) such that $n \sin\theta$ is finite, equal (say) to z. This was proved by taking Murphy's series for P_n (Todhunter, § 23) In like manner Bessel's functions of higher order are limits of those Laplace's functions to which Todhunter gives the name of Associated Functions. A theorem was found for the general functions corresponding to the relation subsisting between three consecutive Bessel's functions [viz. $\frac{1}{2}z\,\{J_{m-1}(z) + J_{m+1}(z)\} = mJ_m(z)$]; Prof. Cayley stated that the results obtained were very interesting. Mr. S. Roberts gave some results bearing upon his paper read at the December meeting. Prof. Cayley gave an expression for the surface of an ellipsoid communicated to him by Prof. Tait. The Chairman, Profs. Cayley and Tanner, and Mr. Webb spoke upon the subject.

R. TUCKER, M.A., *Hon. Sec.*

ON A CLASS OF DETERMINANTS.

By *J. W. L. Glaisher.*

§ 14. The present paper relates to determinants of the forms

$$\begin{vmatrix} a_1, & 1, & ., & ., \dots \\ a_2, & a_1, & 1, & ., \dots \\ a_3, & a_2, & a_1, & 1, \dots \\ a_4, & a_3, & a_2, & a_1, \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} \text{ and } \begin{vmatrix} a_1-b_1, & 1, & ., & ., \dots \\ a_2-b_2, & a_1, & 1, & ., \dots \\ a_3-b_3, & a_2, & a_1, & 1, \dots \\ a_4-b_4, & a_3, & a_2, & a_1, \dots \\ \dots & \dots & \dots & \dots \end{vmatrix},$$

This is the class of determinants that occur in a previous paper, "Expressions for Laplace's coefficients, Bernoullian and Eulerian numbers, &c. as determinants" (vol. VI. pp. 49–63), and as this may be regarded as a continuation of that paper, the numbering of the paragraphs is made continuous.

§ 15. It is shown in § 10 that

$$\frac{1}{1+a_1x+a_2x^2+a_3x^3+\&c.} = 1 - A_1x + A_2x^2 - A_3x^3 + \&c. \qquad \dots\dots\dots (1),$$

where

$$A_n = \begin{vmatrix} a_1, & 1, & ., & ., & \dots \\ a_2, & a_1, & 1, & ., & \dots \\ a_3, & a_2, & a_1, & 1, & \dots \\ a_4, & a_3, & a_2, & a_1, & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} (n \text{ rows}),$$

and that

$$\frac{1+b_1x+b_2x^2+b_3x^3+\&c.}{1+a_1x+a_2x^2+a_3x^3+\&c.} = 1 - P_1x + P_2x^2 - P_3x^3 + \&c. \qquad \dots\dots\dots (2),$$

where

$$P_n = \begin{vmatrix} a_1-b_1, & 1, & ., & ., & \dots \\ a_2-b_2, & a_1, & 1, & ., & \dots \\ a_3-b_3, & a_2, & a_1, & ., & \dots \\ a_4-b_4, & a_3, & a_2, & a_1, & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} (n \text{ rows}).$$

These are the fundamental formulæ. Of course, in the equations (1) and (2) the sign of x may be changed.

§ 16. Changing the sign of x in (1), we have

$$\frac{1}{1+A_1x+A_2x^2+A_3x^3+\&c.}=1-a_1x+a_2x^2-a_3x^3+\&c.,$$

whence we see that if

$$A_n=\begin{vmatrix} a_1, & 1, & ., & \ldots \\ a_2, & a_1, & 1, & \ldots \\ a_3, & a_2, & a_1, & \ldots \\ \multicolumn{4}{c}{\ldots\ldots\ldots\ldots\ldots} \end{vmatrix} \text{ (n rows),}$$

then

$$a_n=\begin{vmatrix} A_1, & 1, & ., & \ldots \\ A_2, & A_1, & 1, & \ldots \\ A_3, & A_2, & A_1, & \ldots \\ \multicolumn{4}{c}{\ldots\ldots\ldots\ldots\ldots} \end{vmatrix} \text{ (n rows).}$$

§ 17. Let

$$\frac{1}{1+b_1x+b_2x^2+b_3x^3+\&c.}=1-B_1x+B_2x^2-B_3x^3+\&c.,$$

so that

$$B_n=\begin{vmatrix} b_1, & 1, & ., & \ldots \\ b_2, & b_1, & 1, & \ldots \\ b_3, & b_2, & b_1, & \ldots \\ \multicolumn{4}{c}{\ldots\ldots\ldots\ldots\ldots} \end{vmatrix}, \quad b_n=\begin{vmatrix} B_1, & 1, & ., & \ldots \\ B_2, & B_1, & 1, & \ldots \\ B_3, & B_2, & B_1, & \ldots \\ \multicolumn{4}{c}{\ldots\ldots\ldots\ldots\ldots} \end{vmatrix},$$

then

$$\frac{1+b_1x+b_2x^2+b_3x^3+\&c.}{1+a_1x+a_2x^2+a_3x^3+\&c.}=\frac{1-A_1x+A_2x^2-A_3x^3+\&c.}{1-B_1x+B_2x^2-B_3x^3+\&c.}\ldots(3),$$

and therefore by (2)

$$\begin{vmatrix} a_1-b_1, & 1, & ., & \ldots \\ a_2-b_2, & a_1, & 1, & \ldots \\ a_3-b_3, & a_2, & a_1, & \ldots \\ \multicolumn{4}{c}{\ldots\ldots\ldots\ldots\ldots} \end{vmatrix}=(-)^n\begin{vmatrix} B_1-A_1, & 1, & ., & \ldots \\ B_2-A_2, & B_1, & 1, & \ldots \\ B_3-A_3. & B_2, & B_1, & \ldots \\ \multicolumn{4}{c}{\ldots\ldots\ldots\ldots\ldots} \end{vmatrix},$$

each determinant having n rows.

For example, put $n=2$, and the theorem gives

$$a_1^2-a_1b_1-(a_2-b_2)=B_1^2-A_1B_1-(B_2-A_2),$$

viz. $a_1^2-a_1b_1-a_2+b_2=b_1^2-a_1b_1-(b_1^2-b_2)+a_1^2-a_2,$

which is true.

§ 18. It is to be noticed that A_n and P_n being defined as in § 15, then

$$P_n = A_n - b_1 A_{n-1} + b_2 A_{n-2} \dots \pm b_n.$$

This is readily proved either by multiplying (1) by

$$1 + b_1 x + b_2 x^2 + b_3 x^3 + \&c.$$

and comparing the right-hand member with that of (2), or by observing that in the determinant P_n the coefficient of b_r is $\pm A_{n-r}$.

§ 19. Since (3) may be written

$$\frac{1 - B_1 x + B_2 x^2 - B_3 x^3 + \&c.}{1 + a_1 x + a_2 x^2 + a_3 x^3 + \&c.} = \frac{1 - A_1 x + A_2 x^2 - A_3 x^3 + \&c.}{1 + b_1 x + b_2 x^2 + b_3 x^3 + \&c.},$$

it follows that

$$\begin{vmatrix} a_1 + B_1, & 1, & ., & ., \dots \\ a_2 - B_2, & a_1, & 1, & ., \dots \\ a_3 + B_3, & a_2, & a_1, & 1, \dots \\ a_4 - B_4, & a_3, & a_2, & a_1, \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} = \begin{vmatrix} b_1 + A_1, & 1, & ., & ., \dots \\ b_2 - A_2, & b_1, & 1, & ., \dots \\ b_3 + A_3, & b_2, & b_1, & 1, \dots \\ b_4 - A_4, & b_3, & b_2, & b_1, \dots \\ \dots & \dots & \dots & \dots \end{vmatrix},$$

each determinant having n rows.

A corresponding equation, in which a_1 and A_1, b_1 and B_1, &c., are interchanged, also holds good.

§ 20. If.

$$P_n = \begin{vmatrix} a_1 - b_1, & 1, & ., \dots \\ a_2 - b_2, & a_1, & 1, \dots \\ a_3 - b_3, & a_2, & a_1, \dots \\ \dots & \dots & \dots \end{vmatrix}, \quad Q_n = \begin{vmatrix} b_1 - a_1, & 1, & ., \dots \\ b_2 - a_2, & b_1, & 1, \dots \\ b_3 - a_3, & b_2, & b_1, \dots \\ \dots & \dots & \dots \end{vmatrix},$$

then

$$Q_n = (-)^n \begin{vmatrix} P_1, & 1, & ., \dots \\ P_2, & P_1, & 1, \dots \\ P_3, & P_2, & P_1, \dots \\ \dots & \dots & \dots \end{vmatrix}, \quad P_n = (-)^n \begin{vmatrix} Q_1, & 1, & ., \dots \\ Q_2, & Q_1, & 1, \dots \\ Q_3, & Q_2, & Q_1, \dots \\ \dots & \dots & \dots \end{vmatrix},$$

each determinant having n rows.

This result is at once seen to be true, for P_n is defined in § 15, and Q_n is given by the equation

$$\frac{1 + a_1 x + a_2 x^2 + a_3 x^3 + \&c.}{1 + b_1 x + b_2 x^2 + b_3 x^3 + \&c.} = 1 - Q_1 x + Q_2 x^2 - Q_3 x^3 + \&c.,$$

so that

$$\frac{1}{1-P_1x+P_2x^2-P_3x^3+\&c.}=1-Q_1x+Q_2x^2-Q_3x^3+\&c.$$

As an example, put $n=2$, and the theorem gives

$$\begin{vmatrix} b_1-a_1, & 1 \\ b_2-a_2, & b_1 \end{vmatrix} = \begin{vmatrix} P_1, & 1 \\ P_2, & P_1 \end{vmatrix} = (a_1-b_1)^2 - \begin{vmatrix} a_1-b_1, & 1 \\ a_2-b_2, & a_1 \end{vmatrix}.$$

§ 21. Other relations that follow at once by means of (2) are

(i) $b_n = a_n - P_1a_{n-1} + P_2a_{n-2} - \ldots \pm P_n$.

(ii) $a_n = \begin{vmatrix} P_1+b_1, & 1, & ., & \ldots \\ P_2-b_2, & P_1, & 1, & \ldots \\ P_3+b_3, & P_2, & P_1, & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}$ (n rows).

(iii) $A_n = \begin{vmatrix} b_1+P_1, & 1, & ., & \ldots \\ b_2-P_2, & b_1, & 1, & \ldots \\ b_3+P_3, & b_2, & b_1, & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}$ (n rows).

§ 22. Since

$$\frac{1}{1-B_1x+B_2x^2-B_3x^3+\&c.}=1+b_1x+b_2x^2+b_3x^3+\&c.,$$

it follows that

$$\frac{1}{1+M_1x+M_2x^2+M_3x^3+\&c.}=\frac{1+b_1x+b_2x^2+b_3x^3+\&c.}{1+a_1x+a_2x^2+a_3x^3+\&c.},$$

where

$$\begin{aligned} M_1 &= a_1 - B_1, \\ M_2 &= a_2 - a_1B_1 + B_2, \\ M_3 &= a_3 - a_2B_1 + a_1B_2 - B_3, \\ &\ldots\ldots\ldots\ldots\ldots\ldots, \end{aligned}$$

so that

$$\begin{vmatrix} a_1-b_1, & 1, & ., & \ldots \\ a_2-b_2, & a_1, & 1, & \ldots \\ a_3-b_3, & a_2, & a_1, & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} = \begin{vmatrix} M_1, & 1, & ., & \ldots \\ M_2, & M_1, & 1, & \ldots \\ M_3, & M_2, & M_1, & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix},$$

each determinant having n rows.

§ 23. Starting with the equation

$$\frac{1}{1-a_1x+a_2x^2-a_3x^3+\&c.}=1+A_1x+A_2x^2+A_3x^3+\&c.,$$

and multiplying by $1-qx+q^2x^2-\&c.$, and also by its equivalent $\dfrac{1}{1+qx}$, we find that

$$\frac{1-qx+q^2x^2-q^3x^3+\&c.}{1-a_1x+a_2x^2-a_3x^3+\&c.}$$

$$=\frac{1}{1-(a_1-q)x+(a_2-qa_1)x^2-(a_3-qa_2)x^3+\&c.}$$

$$=1+(A_1-q)x+(A_2-qA_1+q^2)x^2+(A_3-qA_2+q^2A_1-q^3)x^3+\&c.$$

$$=1+C_1x+C_2x^2+C_3x^3+\&c. \text{ suppose.}$$

Thus we have

$$\begin{vmatrix} a_1-q, & 1, & ., & ., \dots \\ a_2-q^2, & a_1, & 1, & ., \dots \\ a_3-q^3, & a_2, & a_1, & 1, \dots \\ a_4-q^4, & a_3, & a_2, & a_1, \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} \text{ (}n\text{ rows)}$$

$$=\begin{vmatrix} a_1-q, & 1, & ., & ., \dots \\ a_2-qa_1, & a_1-q, & 1, & ., \dots \\ a_3-qa_2, & a_2-qa_1, & a_1-q, & 1, \dots \\ a_4-qa_4, & a_3-qa_2, & a_2-qa_1, & a_1-q, \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} \text{ (}n\text{ rows)}$$

$$=A_n-qA_{n-1}+q^2A_{n-2}\dots\pm q^n, =C_n:$$

and also $$A_n=C_n+qC_{n-1}.$$

The equality of the determinants is easily proved by multiplying the first row of the second determinant by q and adding it to the second row, multiplying the second row of the determinant so formed by q, and adding it to the third row, and so on.

§ 24. As an example, consider the series in § 2, viz.

$$\frac{x}{\log(1+x)}=1+V_1x+V_2x^2+V_3x^3+\&c.,$$

where

$$V_n=\begin{vmatrix} \frac{1}{2}, & 1, & ., \dots \\ \frac{1}{3}, & \frac{1}{2}, & 1, \dots \\ \frac{1}{4}, & \frac{1}{3}, & \frac{1}{2}, \dots \\ \dots & \dots & \dots \end{vmatrix} \text{ (}n\text{ rows).}$$

Let $\dfrac{x}{(1+x)\log(1+x)} = 1 + C_1x + C_2x^2 + C_3x^3 + \&c.,$

so that, dividing by x, and integrating

$$\log(1+x) = \log x + C_1x + \tfrac{1}{2}C_2x^2 + \tfrac{1}{3}C_3x^3 + \&c.$$

Then, by § 23, since $q = 1$,

$$C_n = -\begin{vmatrix} \frac{1}{2}, & 1, & ., & ., & \ldots \\ \frac{2}{3}, & \frac{1}{2}, & 1, & ., & \ldots \\ \frac{3}{4}, & \frac{1}{3}, & \frac{1}{2}, & 1, & \ldots \\ \frac{4}{5}, & \frac{1}{4}, & \frac{1}{3}, & \frac{1}{2}, & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix} \ (n \text{ rows}),$$

and also $= (-)^n \begin{vmatrix} \frac{1}{1.2}, & -1, & ., & ., & \ldots \\ \frac{1}{2.3}, & \frac{1}{1.2}, & -1, & , & \ldots \\ \frac{1}{3.4}, & \frac{1}{2.3}, & \frac{1}{1.2}, & -1, & \ldots \\ \frac{1}{4.5}, & \frac{1}{3.4}, & \frac{1}{2.3}, & \frac{1}{1.2}, & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix} \ (n \text{ rows}).$

Further $\qquad C_n = V_n - V_{n-1} + V_{n-2} \ldots \pm 1,$

and $\qquad V_n = C_n + C_{n-1}.$

NOTE ON MR. LEUDESDORF'S THEOREM IN KINEMATICS.

By *A. B. Kempe, B.A.*

THERE are two points in Mr. Leudesdorf's paper, p. 125, which should, I think, be noticed.

(1) It is not necessary that curves traced out by the points A, B, C, P should all be closed curves in the usual sense in which the term is used, that is, visibly closed curves. It is enough if the points return together to their starting places. This either involves their passing round visibly closed curves or their reciprocating on any curves. An examination of points on the moving plane close to the reciprocating points shows that the latter really describe closed curves, of which the width vanishes, and consequently the area also.

Thus in the case of the three-bar motion referred to at the end of Mr. Leudesdorf's paper, in certain cases the radial bars r, s do not completely rotate, but reciprocate; in this case the areas of the curves described by the points A and B are not πr^2, πs^2 respectively, but vanish.

(2) The integration of θ is not necessarily from 0 to 2π, but may be from 0 to any angle and back to 0, so that the θ term vanishes, and the equation (5) takes the simpler form

$$(P) = x(A) + y(B) + z(C).$$

The following example illustrates the two modifications. Consider the curves described by points on the connecting rod of a beam engine. Here θ does not pass through 2π but returns to 0; also the area of the curve described by that point of the connecting rod which is connected with the beam, is 0. Thus, if r = radius of the crank, $(P) = k.\pi r^2$, where k = ratio in which P divides the distance between the beam pin and the crank pin.

The formula of Mr. Leudesdorf may be arrived at in the following very simple manner.

Let P be connected with A, B, C by the straight lines a', b', c', and let

$$B\hat{P}C = \alpha',$$

$$C\hat{P}A = \beta',$$

$$A\hat{P}B = \gamma';$$

then $$\alpha' + \beta' + \gamma' = 2\pi.$$

Now any small displacement that the system admits of may be made by rotating it about P and displacing it parallel to itself.

Let δs be the linear displacement of P, and let it make an angle σ with PA.

Let the displacements of PA, PB, PC, normal to their directions, be δn_1, δn_2, δn_3, then

$$\delta n_1 = \delta s \sin \sigma,$$

$$\delta n_2 = \delta s \sin(\gamma' + \sigma),$$

$$\delta n_3 = \delta s \sin(\beta' - \sigma);$$

therefore, since $$\alpha' + \beta' + \gamma' = 2\pi,$$

$$\delta n_1 \sin\alpha' + \delta n_2 \sin\beta' + \delta n_3 \sin\gamma' = 0;$$

therefore $\qquad n_1 \sin\alpha' + n_2 \sin\beta' + n_3 \sin\gamma' = 0,$

where n_1, n_2, n_3 are the total normal displacements.

If the system starts from one position and returns to it again, there not being a complete rotation about P, but a partial rotation and a return,

$$(A) = (P) - a'n_1,\ (B) = (P) - b'n_2,\ (C) = (P) - c'n_3;$$

therefore $\quad \dfrac{(A)}{a'}\sin\alpha' + \dfrac{(B)}{b'}\sin\beta' + \dfrac{(C)}{c'}\sin\gamma'$

$$= (P)\left[\frac{\sin\alpha'}{a'} + \frac{\sin\beta'}{b'} + \frac{\sin\gamma'}{c'}\right].$$

This may be put into the form

$$(P) = (A)\,x + (B)\,y + (C)\,z.$$

If the system rotates once the extra term will be added. There is no difficulty in obtaining the modified formula.

Western Circuit,
March 12*th*, 1878.

ON THE RING OF CIRCLES TOUCHING TWO CIRCLES, AND KINDRED PORISMS.

By *W. W. Taylor, M.A.*

In the February number of the *Messenger* (p. 148) it has been proved by my brother, Mr. H. M. Taylor, that the condition for a ring of circles each touching its next neighbours on either side and every one touching each of two primary circles is *when one of these two circles is included within the other*,

$$mc^2 = (a - mb)(am - b) \quad\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where a and b are the radii of the two circles, and c is the distance between their centres, and

$$m = \frac{1 + \sin\frac{\pi}{n}}{1 - \sin\frac{\pi}{n}},$$

where n is *rational, but not necessarily* integral.

Fractional values of n should be reduced to their lowest terms; the numerator then indicates the number of circles that compose a ring, and the denominator the number of complete cycles these circles make with respect to the two original circles.

If one circle is external to the other the sign of b must be changed in the above formula, as is easily seen by writing

down the original equations (1), (2), (3), (4) in the paper referred to (p. 148), in the case where the point of inversion lies between the two concentric circles.

Hence, when a and b are equal, the condition becomes

$$mc^2 = a^2(1+m)^2,$$

or

$$\sqrt{(m)}\, c = a(1+m).$$

Now

$$m = \frac{1+\sin\frac{\pi}{n}}{1-\sin\frac{\pi}{n}} = \frac{\cos^2\frac{\pi}{n}}{\left(1-\sin\frac{\pi}{n}\right)^2};$$

therefore

$$c = a\frac{2}{1-\sin\frac{\pi}{n}} \times \frac{1-\sin\frac{\pi}{n}}{\cos\frac{\pi}{n}} = \frac{2a}{\cos\frac{\pi}{n}} \quad \ldots\ldots\ldots\ldots(2).$$

Now if A and B are two of a ring of p equal circles, such as are described above, circumscribing another circle C, then

$$\frac{a}{c} = \frac{\sin\frac{\pi}{p}}{2\sin\frac{(r+1)\pi}{p}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

where r is the number of circles in the ring intervening between A and B, a and c retaining the same values as above.

The condition for the identity of equations (2) and (3) is

$$\frac{\sin\frac{\pi}{p}}{\sin\frac{(r+1)\pi}{p}} = \cos\frac{\pi}{n} \quad \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Now if p is an even number, we can satisfy this equation by putting $r+1=\frac{1}{2}p$, and finding a relation between p and n from the equation

$$\sin\frac{\pi}{p} = \cos\frac{\pi}{n} = \sin\left(\frac{\pi}{2}-\frac{\pi}{n}\right),$$

or therefore

$$\frac{1}{n}+\frac{1}{p}=\frac{1}{2},$$

this being the *only* solution, because both n and p must be greater than 2.

Equation (4) can also admit of the same solution when p is an odd number, by supplying a circle which shall be

opposite to A, which can be done by the following construction.

Let C and D (fig. 12) be the two original circles and A one of the ring of circles touching them; describe a circle to touch A and C at the point of contact a and also to touch D at b; describe a circle B to touch D at b and also to touch C. A and B will be the required pair. This proves the truth of the following porism.

If two *concentric* circles form such a pair as to admit of a ring of circles being described to touch each member of the ring and the circle adjoining on each side, then any opposite pair of any one of such rings will also fulfil the same condition.

The number of circles in the two rings being connected by the formula

$$\frac{1}{n}+\frac{1}{p}=\frac{1}{2},$$

in which fractional values of n and p must be interpreted as above.

Upon inversion the only alteration that this porism undergoes is that the word *concentric* may be omitted.

From the formula

$$\frac{1}{n}+\frac{1}{p}=\frac{1}{2},$$

we can obtain any number of pairs of values of n and p; it is easily seen, by putting the formula into the form

$$p=2+\frac{4}{n-2},$$

that the only pairs of which both members are integral are

$$n=3,\ p=6,$$

$$n=4,\ p=4,$$

$$n=6,\ p=3.$$

It is curious that when $n=4$ and $p=4$ the same figure (fig. 13) represents both systems of circles, when any pair that are not in contact are taken as the two primary circles.

The other figures (figs. 14—17) represent the cases when

$$n=8,\ p=\tfrac{8}{3},$$

$$n=\tfrac{8}{3},\ p=8.$$

In these the points of contact of successive circles are marked with the numbers 1...8, in order to show how

many cycles are completed; and the circles that have been chosen from the first series to form the nucleus of the second system, have been marked with darker lines.

It is hardly necessary to remark that n and p are interchangeable.

Now let us consider an anchor ring containing a ring of n spheres, each sphere touching the anchor ring along a circle, and also touching the consecutive spheres on either side; the principal transverse section of this will give a figure such as (14, 16); and any axial section will give a figure such as (15, 17), the radii and distances of the centres of the two circles being related to one another in such a manner that there will be a ring of p circles where $\frac{1}{n} + \frac{1}{p} = \frac{1}{2}$; or, considering the symmetry of the figure, there will be a ring of p spheres touching the anchor ring externally along a circle, and each touching one another.

Conversely, if there be a ring of p spheres touching externally, there will be a ring of n spheres touching internally.

We are now in a position to prove the following porism.

If in any binodal cyclide a ring of n spheres can be described, each sphere touching the two adjacent ones, and also touching the cyclide along a circle, then any number of such rings of spheres can be described starting with any sphere that so touches the cyclide, whether internally or externally, the number of spheres in every ring of one set (internal or external according as first ring was internal or external), being n and the number of spheres in every ring of the other set being p $\left(\text{where } \frac{1}{n} + \frac{1}{p} = \frac{1}{2} \text{ as before}\right)$.

To prove this, we have only to take a pair of spheres touching each other, and each touching the cyclide along a circle on the side opposite to that on which the ring of spheres is situated, and invert with respect to the point of contact of these two spheres; the cyclide is thus inverted into an anchor ring with an internal ring of spheres, and the proposition has been proved for such an anchor ring, and therefore on re-inversion we see that it is true for the cyclide.

The condition for the anchor ring is clearly the same, as for the original ring of circles in my brother's paper; therefore on inversion, the condition for the cyclide will be the same as his condition $mc^2 = (a - bm)(am - b)$, paying regard to proper signs of a and b, if for a and b we take the values of the radii of the greatest and least of either set of generating circles.

ON LONG SUCCESSIONS OF COMPOSITE NUMBERS.

By *J. W. L. Glaisher.*

(Continued from p. 106).

§4. THIS section contains lists of sequences of 99 or more consecutive composite numbers occurring in the seventh, eighth, and ninth millions. They were formed in exactly the same manner as those in §2 (pp. 104, 105), viz. all the instances in which the enumeration showed that there were 0, 1, 2, or 3 primes in a century, were looked out in Dase's tables, and the cases noted down in which the sequence was about 50 or upwards. From these, which were very numerous, all the sequences of 99 and upwards were selected, thus forming the following lists:

SEVENTH MILLION.—*Sequences of* 99 *and upwards.*

Lower Limit.	Upper Limit.	Sequence.
6,012,899	6,013,013	113
6,027,283	6,027,383	99
6,034,247	6,034,393	145
6,084,977	6,085,103	125
6,085,441	6,085,561	119
6,158,563	6,158,681	117
6,242,263	6,242,363	99
6,333,799	6,333,907	107
6,347,059	6,347,177	117
6,371,401	6,371,537	135
6,376,193	6,376,303	109
6,385,619	6,385,733	113
6,429,223	6,429,323	99
6,613,631	6,613,753	121
6,646,049	6,646,153	103
6,655,423	6,655,531	107
6,726,821	6,726,947	125
6,752,623	6,752,747	123
6,789,793	6,789,901	107
6,808,273	6,808,397	123
6,812,233	6,812,339	105
6,826,159	6,826,271	111
6,836,867	6,836,969	101
6,845,417	6,845,519	101
6,851,863	6,851,963	99
6,877,109	6,877,219	109
6,879,683	6,879,797	113

Lower Limit.	Upper Limit.	Sequence.
6,958,667	6,958,801	133
6,972,127	6,972,233	105
6,983,843	6,983,947	103
6,991,147	6,991,253	105

EIGHTH MILLION.—*Sequences of* 99 *and upwards.*

Lower Limit.	Upper Limit.	Sequence.
7,129,877	7,130,003	125
7,130,579	7,130,687	107
7,160,227	7,160,347	119
7,227,551	7,227,667	115
7,230,331	7,230,479	147
7,293,989	7,294,093	103
7,309,427	7,309,529	101
7,345,967	7,346,069	101
7,445,047	7,445,159	111
7,494,763	7,494,869	105
7,565,191	7,565,303	111
7,621,259	7,621,399	139
7,629,371	7,629,487	115
7,662,517	7,662,617	99
7,683,131	7,683,233	101
7,702,573	7,702,687	113
7,743,233	7,743,371	137
7,753,679	7,753,787	107
7,771,307	7,771,411	103
7,784,039	7,784,159	119
7,803,491	7,803,611	119
7,826,899	7,827,019	119
7,881,373	7,881,487	113
7,950,001	7,950,109	107

NINTH MILLION.—*Sequences of* 99 *and upwards.*

Lower Limit.	Upper Limit.	Sequence.
8,001,359	8,001,491	131
8,027,699	8,027,807	107
8,073,647	8,073,749	101
8,166,107	8,166,209	101
8,172,713	8,172,821	107
8,208,449	8,208,553	103
8,211,227	8,211,331	103
8,294,021	8,294,123	101
8,303,957	8,304,061	103
8,332,427	8,332,529	101
8,350,483	8,350,583	99
8,367,397	8,367,517	119

Lower Limit.	Upper Limit.	Sequence.
8,409,721	8,409,829	107
8,421,251	8,421,403	151
8,441,869	8,441,969	99
8,454,959	8,455,063	103
8,470,927	8,471,053	125
8,487,421	8,487,527	105
8,514,949	8,515,063	113
8,648,557	8,648,677	119
8,752,871	8,752,987	115
8,793,881	8,793,991	109
8,806,891	8,806,991	99
8,841,529	8,841,629	99
8,850,671	8,850,773	101
8,889,031	8,889,143	111
8,905,199	8,905,321	121
8,917,523	8,917,663	139
8,939,597	8,939,699	101
8,947,217	8,947,319	101
8,981,461	8,981,563	101

It will be noticed that the two longest sequences in these three millions are 151, in the ninth million, and 147, in the eighth million, while a sequence of 147 was met with at the beginning of the third million (2,010,733—2,010,881). The longest sequence found in the first million was 113, in the second 131, in the third 147, in the seventh 145, in the eighth 147, in the ninth 151, so that the lists for the seventh and eighth millions do not contain longer sequences than had already been found in the third million, and the ninth million only produces an extension of 2. Attention, however, should be directed to the remarks in § 2 (p. 105), where it is pointed out that the method adopted does not give all the sequences, so that the lists are necessarily incomplete. A sequence exceeding 151 might, of course, escape detection if each of the centuries over which it extended contained more than 3 primes; but even supposing each of the centuries to contain only 4 primes, it is clear that a sequence exceeding about 170 would be impossible, and probably a lower limit could be fixed. The sequence of 151 was obtained from a 0-prime century, while that of 147 was obtained both from a 3-prime and a 2-prime century, there being 3 primes in the century 7,230,300—7,230,400, and 2 in the century 7,230,400—7,230,500.

§ 5. Having by me in MS. a *complete* list of primes, with their differences, up to about 30,000 (which was formed some

time since for a special purpose), I selected all the instances in which the difference between two consecutive primes was equal to or exceeded 20 (*i.e.* in which the sequence was equal to or exceeded 19), so as to obtain the relatively long sequences that occur early in the series of natural numbers. It was subsequently found on trial to be almost as easy to pick out the sequences from Chernac's *Cribrum Arithmeticum* as from the MS., and I accordingly had all the sequences of 19 and upwards looked out from Chernac up to 100,000.* The results are contained in the following table:

1—100,000.—*Numbers of Sequences of* 19 *and upwards.*

Length of Sequence.	Number of Sequences.	Length of Sequence.	Number of Sequences.
19	238	43	5
21	223	45	4
23	206	47	3
25	88	49	5
27	98	51	7
29	146	53	4
31	32	55	1
33	33	57	4
35	54	59	1
37	19	61	1
39	28	63	1
41	19	71	1

Total number of sequences = 1,221.

The most remarkable feature of this table is that it shows that there are no less than 146 sequences of 29, although there are only 98 sequences of 27, and 32 sequences of 31. It seemed at first that there must have been some error in the work; but the whole enumeration was performed again, and with the same result. I have myself looked out the 146 sequences of 29 and verified them.†

The meaning of the next list is as follows: between 11 and 23 there is no sequence exceeding 3 till 5 occurs at 23–29; the first sequence that exceeds 5 is a sequence of 7 at 89–97, the first sequence that exceeds 7 is a sequence of 13 at 113–127, and so on. Thus between 19,661 and 31,397 there is no sequence that exceeds 51, but, then, the longest sequence, 71, occurs. Perhaps the most remarkable of these

* One half-column (68,303—68,399) in Chernac is erroneous, in consequence of a bar having fallen out from the bottom and been replaced at the top, as was pointed out by Burckhardt in the preface to his first million. Account has been taken of this in the enumeration of the sequences.

† Since this was written a third enumeration, from Burckhardt's tables, has been made. The result entirely confirms the above table.

sequences is that of 33 which occurs as early as 1,327–1,361, for beyond this no sequence so long as 33 is met with until 8,467–8,501, where there is another sequence of 33; but there are 10 sequences of 29 in this interval.

Lower Limit.	Upper Limit.	Sequence.
7	11	3
23	29	5
89	97	7
113	127	13
523	541	17
887	907	19
1,327	1,361	33
9,551	9,587	35
15,683	15,727	43
19,609	19,661	51
31,397	31,469	71

The following table contains the sequences of 51 and upwards that occur in the first 100,000 natural numbers.

1—100,000.—*Sequences of* 51 *and upwards.*

Lower Limit.	Upper Limit.	Sequence.
19,609	19,661	51
25,471	25,523	51
31,397	31,469	71
34,061	34,123	61
35,617	35,671	53
35,677	35,729	51
40,289	40,343	53
40,639	40,693	53
43,331	43,391	59
43,801	43,853	51
44,293	44,351	57
48,679	48,731	51
58,831	58,889	57
59,281	59,333	51
74,959	75,011	51
79,699	79,757	57
82,073	82,129	55
85,933	85,991	57
86,869	86,923	53
89,689	89,753	63

It should be specially noted that the method adopted for obtaining the sequences given in this section (viz. those between 1 and 100,000) should yield *all* the sequences, so

that if there are any omissions, they are due to errors. The work was performed in duplicate, and is, I believe, correct, but I have only verified a portion of it myself.

§6. In §1 "the hundred numbers between $100n$ and $100(n+1)$" are defined as the $(n+1)^{\text{th}}$ century. The words between inverted commas should be "the hundred numbers between $100n-1$ and $100(n+1)$"; so that, for example, the third century consists of the numbers 200, 201, ...299, while the first century only consists of the 99 numbers 1, 2, ...99 (unless 0 be considered a number). Of course, in matters relating to the enumeration of primes, it is a matter of indifference whether the first century be defined to be 1, 2, ...100, the second 101, 102, ...200 and so on, or whether the first be (0), 1, 2, ...99, the second 100, 101, ...199 and so on, as the numbers 100, 200, &c. are even numbers and do not appear in the factor tables at all, so that practically, a century consists of the 99 numbers between $100n$ and $100(n+1)$. As a rule, in the arrangement of tables, it is more convenient to adopt the latter system, so that all the numbers of the same century may have the same figure in the hundreds' place.*

§7. There are many instances near the beginning of the series of natural numbers of two primes being separated by only one number, such as 29 and 31, 41 and 43, 59 and 61, &c. These may be called prime-pairs. A glance at Burckhardt's and Dase's tables shows that prime-pairs are of tolerably frequent occurrence at all parts of the tables, although they become less numerous as we advance higher in the series of natural numbers. To exhibit this diminution I have had all the prime-pairs that occur in each chiliad of the first hundred chiliads of each of the six millions counted and arranged in tables. These are complete and will form a separate paper.

* The same uncertainty exists with regard to a century of years, viz., as to whether the nineteenth century began on January 1, 1800, or January 1, 1801. It is only a question of usage, but there is no ancient usage, as the term century, in this sense, is modern. "A century is any collection of one hundred; its restriction to collection of years is modern...We hold it clear that no usage can exist except one of very modern times. The present practice of astronomers and chronologers is to make the first year of the reckoning to be the first year of a century, so that A.D. 1—100 is the first century, A.D. 1801—1900 is the nineteenth century." (De Morgan, *Companion to the Almanac*, 1850, pp. 24—26). In tabular matter, however, where either 0 or 1 may be taken as the starting point, the former seems more adapted to our system of numeration. The tendency to begin from 1 is probably a relic of the Roman system i, ii, iii, &c., in which zero has no place as the commencement of the series of numerals.

FORMULÆ INVOLVING THE SEVENTH ROOTS OF UNITY.

By Professor *Cayley*.

LET ω be an imaginary cube root of unity, $\omega^2+\omega+1=0$, or say $\omega=\frac{1}{2}\{-1+i\sqrt{(3)}\}$; $\alpha^3=-7(1+3\omega)$, $\beta^3=-7(1+3\omega^2)$ (values giving $\alpha^3\beta^3=343$), and the cube roots α, β being such that $\alpha\beta=7$; then $\alpha+\beta, =\alpha+\frac{7}{\alpha}$, is a three-valued function $\left(\text{since changing the root } \omega \text{ we merely interchange } \alpha \text{ and } \frac{7}{\alpha}\right)$; and if r be an imaginary seventh root of unity, then

$$3(r+r^6)=\alpha+\beta-1,$$
$$3(r^2+r^5)=\omega\alpha+\omega^2\beta-1,$$
$$3(r^4+r^3)=\omega^2\alpha+\omega\beta-1.$$

Any one of these formulæ gives the other two; for observe that we have $\alpha^3=-\alpha\beta(1+3\omega)$, $\beta^3=-\alpha\beta(1+3\omega^2)$, that is $\alpha^2=-\beta(1+3\omega)$, $\beta^2=-\alpha(1+3\omega^2)$; hence, starting for instance with the first formula, we deduce

$$\begin{aligned}9(r^2+r^5+2) &= \alpha^2+2\alpha\beta+\beta^2-2\alpha-2\beta+1,\\ &=-\beta(1+3\omega)+14-\alpha(1+3\omega^2)-2\alpha-2\beta+1,\\ &=-\alpha(3+3\omega^2)-\beta(3+3\omega)+15,\\ &= 3\omega\alpha+3\omega^2\beta+15,\end{aligned}$$

that is

$$3(r^2+r^5)=\omega\alpha+\omega^2\beta-1,$$

and in like manner by squaring each side of this we have the third formula

$$3(r^4+r^3)=\omega^2\alpha+\omega\beta-1.$$

The foregoing formulæ apply to the combinations $r+r^6$, r^2+r^5, r^4+r^3 of the seventh roots of unity, but we may investigate the theory for the roots themselves r, r^2, r^3, r^4, r^5, r^6. These depend on the new radical $\sqrt{(-7)}$ or $i\sqrt{(7)}$; introducing instead hereof X, Y, where

$$X=\tfrac{1}{2}\{-1+i\sqrt{(7)}\},$$
$$Y=\tfrac{1}{2}\{-1-i\sqrt{(7)}\},$$

then if
$$A^3 = 6 + 3\omega X + (1 + 3\omega^2) Y,$$
$$B^3 = 6 + 3\omega^2 X + (1 + 3\omega) Y,$$
where
$$AB = i\sqrt{(7)},$$
we have (Lagrange, *Equations Numeriques*, p. 294),
$$3r = X + A + B,$$
and I found that in order to bring this into connexion with the foregoing formula, $3(r + r^6) = \alpha + \beta - 1$, where as before $\alpha^3 = -7(1 + 3\omega)$, $\beta^3 = -7(1 + 3\omega^2)$, $\alpha\beta = 7$, it was necessary that B, A should be linear multiples of α, β respectively, the coefficients being rational functions of ω, X; and that the actual relations were
$$B = \frac{\alpha}{7}\{4 - \omega + X(1 - 2\omega)\},$$
$$A = \frac{\beta}{7}\{5 + \omega + X(3 + 2\omega)\};$$
in verification of which, it may be remarked that these equations give
$$AB = \frac{\alpha\beta}{49}\{(20 - \omega - \omega^2) + X(17 - 4\omega - 4\omega^2) + X^2(3 - 4\omega - 4\omega^2)\},$$
viz. in virtue of the equation $\omega^2 + \omega + 1 = 0$, the term in $\{\ \}$ is $= 21 + 21X + 7X^2$, $= 7(X^2 + 3X + 3)$, or since $X^2 + X + 2 = 0$, this is $= 7(2X + 1)$, $= 7i\sqrt{(7)}$, the equation thus is $7AB = \alpha\beta . i\sqrt{(7)}$, which is true in virtue of $AB = i\sqrt{(7)}$ and $\alpha\beta = 7$. The same relations may also be written
$$-\alpha = B(\omega^2 + X)$$
$$-\beta = A(\omega + X).$$
I found in the first instance
$$3r = X + A + B,$$
$$3r^6 = -1 - X + A(\omega^2 - X) + B(\omega - X),$$
$$3r^2 = X + \omega^2 A + \omega B,$$
$$3r^5 = -1 - X + A(\omega - \omega^2 X) + B(\omega^2 - \omega X),$$
$$3r^4 = X + \omega A + \omega^2 B,$$
$$3r^3 = -1 - X + A(1 - \omega X) + B(1 - \omega^2 X),$$
which in fact gave the foregoing formulæ
$$3(r + r^6) = -1 + \alpha + \beta,$$
$$3(r^2 + r^5) = -1 + \omega\alpha + \omega^2\beta,$$
$$3(r^4 + r^3) = -1 + \omega^2\alpha + \omega\beta.$$

But there is a want of symmetry in these expressions for r, r^2, &c., inasmuch as the values of r, r^2, r^4 are of a different form from those of r^6, r^5, r^3; to obtain the proper forms we must for A, B substitute their values in terms of α, β, and we thus obtain

$$3r = \quad X + \frac{\alpha}{7}\{\ 4 - \ \omega + X(\ 1-2\omega)\} + \frac{\beta}{7}\{\ 5+\ \omega + X(\ 3+2\omega)\},$$

$$3r^6 = -1 - X + \frac{\alpha}{7}\{\ 3+\ \omega + X(-1+2\omega)\} + \frac{\beta}{7}\{\ 2-\ \omega + X(-3-2\omega)\},$$

$$3r^2 = \quad X + \frac{\alpha}{7}\{\ 1+5\omega + X(\ 2+3\omega)\} + \frac{\beta}{7}\{-4-5\omega + X(-1-3\omega)\},$$

$$3r^5 = -1 - X + \frac{\alpha}{7}\{-1+2\omega + X(-2-3\omega)\} + \frac{\beta}{7}\{-3-2\omega + X(\ 1+3\omega)\},$$

$$3r^4 = \quad X + \frac{\alpha}{7}\{-5-4\omega + X(-3-\ \omega) + \frac{\beta}{7}(-1+4\omega) + X(-2+\ \omega)\},$$

$$3r^3 = -1 - X + \frac{\alpha}{7}\{-2-3\omega + X(\ 3+\ \omega)\} + \frac{\beta}{7}\{\ 1+3\omega + X(\ 2-\ \omega)\},$$

viz. each of the imaginary seventh roots is thus expressed as a linear function of the cubic radicals α, β (involving ω under the radical signs) with coefficients which are functions of ω, X.

Recollecting the equations $\alpha^2 = -\beta(1+3\omega)$, $\beta^2 = -\alpha(1+3\omega)$, $\alpha\beta = 7$; $\omega^2+\omega+1=0$, $X^2+X+2=0$; it is clear that, starting for instance from the equation for $3r$, and squaring each side of the equation, we should, after proper reductions, obtain for $9r^2$ an expression of the like form; viz. we thus in fact obtain the expression for $3r^2$; then from the expressions of $3r$ and $3r^2$, multiplying together and reducing, we should obtain the expression for $3r^3$; and so on, viz. from any one of the six equations we can in this manner obtain the remaining five equations.

At the time of writing what precedes I did not recollect Jacobi's paper "Ueber die Kreistheilung und ihre Anwendung auf die Zahlentheorie, *Berliner Monatsber.*, b. 1837 and *Crelle*, t. 30 (1846), pp. 166–182. The starting point is the following theorem: if x be a root of the equation $\dfrac{x^p-1}{x-1}=0$, p a prime number, then if g is a prime root of p, and

$$F(\alpha) = x + \alpha x^g + \alpha^2 x^{g^2} \ldots + \alpha^{p-1} x^{g^{p-2}},$$

N 2

where α is any root of $\dfrac{\alpha^{p-1}-1}{\alpha-1}=0$, we have

$$F(\alpha^m)\,F(\alpha^n)=\psi(\alpha)\,F(\alpha^{m+n}),$$

where $\psi(\alpha)$ is a rational and integral function of α with integral coefficients; or, what is the same thing, if α, β be any two roots of the above-mentioned equation, then

$$F(\alpha)\,F(\beta)=\psi(\alpha,\beta)\,F(\alpha\beta),$$

where $\psi(\alpha,\beta)$ is a rational and integral function of α, β with integral coefficients; as regards the proof of this it may be remarked that writing x^g for x, $F(\alpha)$, $F(\beta)$, and $F(\alpha\beta)$ become respectively $\alpha^{-1}F(\alpha)$, $\beta^{-1}F(\beta)$, $(\alpha\beta)^{-1}F(\alpha\beta)$; hence, $F(\alpha)\,F(\beta)\div F(\alpha\beta)$ remains unaltered, and it thus appears that the function in question is expressible rationally in terms of the *adjoint* quantities α and β. With this explanation the following extract will be easily intelligible:

"The true form (never yet given) of the roots of the equation $x^p-1=0$ is as follows: The roots, as is known, can easily be expressed by mere addition of the functions $F(\alpha)$. If λ is a factor of $p-1$ and $\alpha^\lambda=1$, then it is further known that $\{F(\alpha)\}^\lambda$ is a mere function of α. But it is only necessary to know those values of $F(\alpha)$ for which λ is *the power of a prime number*. For suppose $\lambda\lambda'\lambda''\ldots$ is a factor of $p-1$; further let λ, λ', $\lambda''\ldots$ be powers of different prime numbers, and α, α', $\alpha''\ldots$ prime λth, λ'th, λ''th roots of unity, then

$$F(\alpha\alpha'\alpha''\ldots)=\frac{F(\alpha)\,F(\alpha')\,F(\alpha'')\ldots}{\psi(\alpha,\alpha',\alpha''\ldots)}$$

where $\psi(\alpha,\alpha',\alpha''\ldots)$ denotes a rational and integral function of α, α', $\alpha''\ldots$ with integral coefficients. Hence, considering always the $(p-1)$th roots of unity as given, there are contained in the expression for x only radicals the exponents of which are powers of prime numbers, and products of such radicals. But if λ is a power of a prime number, $=\mu^n$, suppose, the corresponding function $F(\alpha)$ can be found as follows: Assume

$$F(\alpha)\,F(\alpha^i)=\psi_i(\alpha)\,F(\alpha^{i+1}),$$

then
$$F(\alpha)=\sqrt[\mu]{\{\psi_1(\alpha)\;\psi_2(\alpha)\ldots\psi_{\mu-1}(\alpha)\,F(\alpha^\mu)\}},$$
$$F(\alpha^\mu)=\sqrt[\mu]{\{\psi_1(\alpha^\mu)\,\psi_2(\alpha^\mu)\ldots\psi_{\mu-1}(\alpha^\mu)\,F(\alpha^{\mu^2})\}},$$

and so on up to

$$F(\alpha^{\mu^{n-1}})=\sqrt[\mu]{\{\psi_1(\alpha^{\mu^{n-1}})\psi_2(\alpha^{\mu^{n-1}})\ldots\psi_{\mu-1}(\alpha^{\mu^{n-1}})\,(-)^{\frac{p-1}{\mu}}p\}},$$

[so that the formulæ contain ultimately μ-th roots only. It is remarked in a foot-note that when $n=1$, the $\mu-1$ functions can always be reduced to one-sixth part in number, and that by an induction continued as far as $\mu=31$, Jacobi had found that all the functions ψ could be expressed by means of the values of a single one of these functions].

The $\mu-1$ functions determine not only the values of all the magnitudes under the radical signs, but also the mutual dependence of the radicals themselves. For replacing α by the different powers of α, one can by means of the values so obtained for these functions rationally express all the μ^n-1 functions $F(\alpha^i)$ by means of the powers of $F(\alpha)$; since all the μ^n-1 magnitudes $\{F(\alpha)\}^i \div F(\alpha^i)$ are each of them equal to a product of several of the functions $\psi(\alpha)$. Herein consists one of the great advantages of the method over that of Gauss, since in this the discovery of the mutual dependency of the different radicals requires a special investigation, which, on account of its laboriousness, is scarcely practicable for even small primes; whereas the introduction of the functions ψ gives simultaneously the quantities under the radical signs, and the mutual dependency of the radicals. The formation of the functions ψ is obtained by a very simple algorithm, which requires only that one should, from the table for the residues of g^m, form another table giving $g^{m'}=1+g^m\,(\text{mod. } p)$, [see table IV. of the Memoir]. According to these rules one of my auditors [Rosenhain] in a Prize-Essay of the [Berlin] *Academy* has completely solved the equations $x^p-1=0$ for all the prime numbers p up to 103."

I am endeavouring to procure the Prize-Essay just referred to. As an example—which however is too simple a one to fully bring out Jacobi's method, and its difference from that of Gauss—consider the equation for the fifth roots of unity, $x^4+x^3+x^2+x+1=0$. According to Gauss, we have $x+x^4$ and x^2+x^3, the roots of the equation $u^2+u-1=0$; say $x+x^4=\frac{1}{2}\{-1+\surd(5)\}$, $x^2+x^3=\frac{1}{2}\{-1-\surd(5)\}$. The first of these combined with $x.x^4=1$ gives $x-x^4=\surd[-\frac{1}{2}\{5+\surd(5)\}]$; and thence $4x=-1+\surd(5)+\surd[-2\{5+\surd(5)\}]$; if from the second of them combined with $x^2.x^3=1$, we were in like manner to obtain the values of x^2 and x^3, it would be necessary to investigate the signs to be given to the radicals, in order that the values so obtained for x^2 and x^3 might be consistent with the value just found for x. For the Jacobian process, observing that a prime fourth root of unity is $\alpha=i$, and writing for shortness F_1, F_2, F_3, F_4 to denote $F(\alpha)$,

$F(\alpha^2)$, $F(\alpha^3)$, $F(\alpha^4)$ respectively, these functions are

$$\begin{aligned}F_1 &= x - x^4 + i(x^2 - x^3),\\ F_2 &= x + x^4 - (x^2 + x^3),\\ F_3 &= x - x^4 - i(x^2 - x^3),\\ F_4 &= x + x^4 + x^2 + x^3,\end{aligned}$$

viz. we have $F_4 = -1$, $F_2^2 = 5$, or say $F_2 = \surd(5)$, $F_1^2 = -(1+2i)F_2$, $= -(1+2i)\surd(5)$; and similarly $F_3^2 = -(1-2i)F_2$, $= -(1-2i)\surd(5)$; but also $F_1F_3 = -5$, so that the values $F_1 = \surd\{-(1+2i)\surd(5)\}$, $F_3 = \surd\{-(1-2i)\surd(5)\}$, must be taken consistently with this last equation $F_1F_3 = \surd(5)$. The values of F_1, F_2, F_3, F_4 being thus known, the four equations then give simultaneously x, x^4, x^2, x^3, these values being of course consistent with each other. It may be remarked that the form in which x presents itself is

$$4x = -1 + \surd(5) + \surd\{-(1+2i)\surd(5)\} + \surd\{-(1-2i)\surd(5)\};$$

with the before-mentioned condition as to the last two radicals; with this condition we in fact have

$$\surd\{-(1+2i)\surd(5)\} + \surd\{-(1-2i)\surd(5)\} = \surd[-2\{5+\surd(5)\}],$$

as is at once verified by squaring the two sides.

NOTE ON HYDRODYNAMICS.

By Professor *E. J. Nanson, M.A.*

THREE first integrals of the ordinary hydrodynamical equations were obtained by Cauchy in the case of an incompressible fluid, and have since been extended by Stokes to the case of any fluid in which the pressure is a function of the density. These integrals may be written in the form

$$\left.\begin{aligned}\frac{\xi}{\rho} &= \frac{\xi_0}{\rho_0}\frac{dx}{da} + \frac{\eta_0}{\rho_0}\frac{dx}{db} + \frac{\zeta_0}{\rho_0}\frac{dx}{dc}\\ \frac{\eta}{\rho} &= \frac{\xi_0}{\rho_0}\frac{dy}{da} + \frac{\eta_0}{\rho_0}\frac{dy}{db} + \frac{\zeta_0}{\rho_0}\frac{dy}{dc}\\ \frac{\zeta}{\rho} &= \frac{\xi_0}{\rho_0}\frac{dz}{da} + \frac{\eta_0}{\rho_0}\frac{dz}{db} + \frac{\zeta_0}{\rho_0}\frac{dz}{dc}\end{aligned}\right\}\text{..............} \quad (\mathrm{I}),$$

and may be found at p. 182 of Besant's *Hydromechanics*, 3rd edition; from them Helmholtz's laws of vortex motion may be deduced. The laws of vortex motion for any fluid in which p is a fraction of ρ may also be deduced from the following equations, which are an extension of those originally given by Stokes and Helmholtz.

$$\left.\begin{aligned}\frac{d}{dt}\left(\frac{\xi}{\rho}\right)=\frac{\xi}{\rho}\frac{du}{dx}+\frac{\eta}{\rho}\frac{du}{dy}+\frac{\zeta}{\rho}\frac{du}{dz}\\ \frac{d}{dt}\left(\frac{\eta}{\rho}\right)=\frac{\xi}{\rho}\frac{dv}{dx}+\frac{\eta}{\rho}\frac{dv}{dy}+\frac{\zeta}{\rho}\frac{dv}{dx}\\ \frac{d}{dt}\left(\frac{\zeta}{\rho}\right)=\frac{\xi}{\rho}\frac{dw}{dx}+\frac{\eta}{\rho}\frac{dw}{dy}+\frac{\zeta}{\rho}\frac{dw}{dz}\end{aligned}\right\}\text{..........(II).}$$

These equation may be at once deduced from the equations I. For, differentiating the first of equations I with respect to t, we have

$$\begin{aligned}\frac{d}{dt}\left(\frac{\xi}{\rho}\right)= &\ \frac{\xi_0}{\rho_0}\frac{du}{da}+\frac{\eta_0}{\rho_0}\frac{du}{db}+\frac{\zeta_0}{\rho_0}\frac{du}{dc}\\ = &\ \frac{\xi_0}{\rho_0}\left\{\frac{du}{dx}\frac{dx}{da}+\frac{du}{dy}\frac{dy}{da}+\frac{du}{dz}\frac{dz}{da}\right\}\\ &+\frac{\eta_0}{\rho_0}\left\{\frac{du}{dx}\frac{dx}{db}+\frac{du}{dy}\frac{dy}{db}+\frac{du}{dz}\frac{dz}{db}\right\}\\ &+\frac{\zeta_0}{\rho_0}\left\{\frac{du}{dx}\frac{dx}{dc}+\frac{du}{dy}\frac{dy}{dc}+\frac{du}{dz}\frac{dz}{dc}\right\}\\ = &\ \frac{\xi}{\rho}\frac{du}{dx}+\frac{\eta}{\rho}\frac{du}{dy}+\frac{\zeta}{\rho}\frac{du}{dz}.\end{aligned}$$

Thomson has, however, given a more general theorem, which includes the above. This theorem is that the value of

$$\int(udx+vdy+wdz)$$

taken round any circuit, moving with the fluid, is constant with regard to the time. If the circuit be infinitely small, and A, B, C denote the areas of its projections on the three coordinate planes, the theorem becomes

$$A\xi+B\eta+C\zeta=A_0\xi_0+B_0\eta_0+C_0\zeta_0\text{....... (III),}$$

or as we may write it

$$\frac{d}{dt}(A\xi+B\eta+C\zeta)=0\text{............... (IV).}$$

Professor H. Lamb has shown that the equations II follow from IV, and *vice versâ*; *suprâ* p. 42.

The equations I may be deduced from III in the following manner.

Let $x+X$, $y+Y$, $z+Z$ be the coordinates of any point on an infinitely small circuit surrounding the point x, y, z, and let $a, b, c, \alpha, \beta, \gamma$, be the values when $t=0$ of x, y, z, X, Y, Z; then

$$X = \frac{dx}{da}\alpha + \frac{dx}{db}\beta + \frac{dx}{dc}\gamma,$$

$$dX = \frac{dx}{da}d\alpha + \frac{dx}{db}d\beta + \frac{dx}{dc}d\gamma,$$

and therefore we have

$$\begin{aligned} 2A &= \int (YdZ - ZdY) \\ &= \frac{d(y,z)}{d(b,c)}\int(\beta d\gamma - \gamma d\beta) \\ &\quad + \frac{d(y,z)}{d(c,a)}\int(\gamma d\alpha - \alpha d\gamma) \\ &\quad + \frac{d(y,z)}{d(a,b)}\int(\alpha d\beta - \beta d\alpha), \end{aligned}$$

or

$$A = \frac{d(y,z)}{d(b,c)}A_0 + \frac{d(y,z)}{d(c,a)}B_0 + \frac{d(y,z)}{d(a,b)}C_0,$$

and similar values are found for B, C. Hence III becomes

$$\begin{aligned} &A_0\left(\frac{d(y,z)}{d(b,c)}\xi + \frac{d(z,x)}{d(b,c)}\eta + \frac{d(x,y)}{d(b,c)}\zeta - \xi_0\right) \\ &+ B_0\left(\frac{d(y,z)}{d(c,a)}\xi + \frac{d(z,x)}{d(c,a)}\eta + \frac{d(x,y)}{d(c,a)}\zeta - \eta_0\right) \\ &+ C_0\left(\frac{d(y,z)}{d(a,b)}\xi + \frac{d(z,x)}{d(a,b)}\eta + \frac{d(x,y)}{d(a,b)}\zeta - \zeta_0\right) = 0, \end{aligned}$$

and since this must hold for any infinitely small circuit surrounding the point x, y, z, the coefficients of A_0, B_0, C_0 in this equation must separately vanish; this, with the equation of continuity,

$$\rho\frac{d(x,y,z)}{d(a,b,c)} = \rho_0$$

gives the equations I. Conversely, from the equation I, III follows at once.

Hence Thomson's theorem, in the case of an infinitely small circuit, stated in the form IV, is precisely equivalent to the Helmholtz differential equations II, and stated in the integral form III, it is precisely equivalent to the Cauchy integrals I.

University, Melbourne,
September 26, 1877.

ON THE INTERPRETATION OF A PASSAGE IN MERSENNE'S WORKS.

By *M. Edouard Lucas.*

M. Genocchi has recently called attention, *àpropos* of a paper of mine, to a passage in Mersenne's Works, from which it results that numbers of the form 2^n-1 are composite, except when n has the values

$$2, 3, 5, 7, 13, 17, 19, 31, 67, 127, 257, \ldots.$$

I may observe that, in order to verify by known methods the last assertion of Mersenne, viz. that $2^{257}-1$ is a prime, the whole population of the globe, calculating simultaneously, would require more than a million of millions of millions of centuries.

Numbers of the form $2^n \pm 1$ can only be prime, with the sign $-$ if the exponent be prime, and with the sign $+$ if the exponent is a power of 2; and it is known that the primes of the latter form are those for which the circumference of a circle may be geometrically divided into equal parts. We have, then, to consider the three distinct classes

$$A=2^{4q+3}-1,\quad B=2^{4q+1}-1,\quad C=2^{2^n}+1.$$

I may add that Fermat has given the decomposition

$$2^{37}-1=223\times 616318177,$$

and Plana has given

$$2^{41}-1=13367\times 164511353.$$

By means of a new method M. Landry has lately found the decompositions

$$2^{43}-1=431\times 9719\times 2099863,$$

$$2^{47}-1=2351\times 4513\times 13264529,$$

$$2^{53}-1=6361\times 69431\times 20394401,$$

$$2^{59}-1=179951\times 3203431780337,$$

and I have myself remarked that

$$2^{73}-1 \equiv 0 \pmod{439},$$

and proved the theorem:—

If the numbers $4q+3$ and $8q+7$ are prime, then

$$2^{4q+3}-1 \equiv 0 \pmod{8q+7};$$

and therefore the numbers

$$2^{83}-1,\ 2^{131}-1,\ 2^{179}-1,\ 2^{191}-1,\ 2^{239}-1,\ 2^{251}-1,\ \ldots$$

are not prime.

En résumé all these results seem to indicate that Mersenne was in possession of arithmetical methods that are now lost. I shall now indicate a new method of verification for each of the forms A and B.

1°. Numbers of the form $A = 2^{4q+3}-1$.

Form the series of numbers

$$1,\ 3,\ 7,\ 47,\ 2207,\ 4870847,\ 27325150497407,\ \ldots,$$

in which each is equal to the square of the preceding one diminished by 2, and retain the residues to modulus A; the calculation of the residues is easily performed by successive subtractions, the first ten multiples of A having been first calculated.

If no one of the $4q+3$ first residues is equal to zero, the number A is composite; if the first zero is comprised within the limits $2q+1$ and $4q+3$, the number A is prime; in fact, if α, $< 2q+1$, denotes the position of the first zero residue, the divisors of A belong to the form $2^{\alpha}k \pm 1$, and to the quadratic form x^2-2y^2.

Example. For $A = 2^7-1$, we have the residues

$$1,\ 3,\ 7,\ 47,\ 48,\ 16,\ 0 \pmod{127},$$

whence the number is prime.

For $A = 2^{11}-1$ we form the residues

$$1,\ 3,\ 7,\ 47,\ 160,\ 1034,\ 620,\ -438,\ -576,\ 160;$$

and A, $=2047$, is not prime and the residues reproduce themselves periodically. Thus $2^{11}-1$ is composite,

$$2^{11}-1 = 23 \times 89.$$

2°. Numbers of the form $B = 2^{4q+1}-1$.

Form the series of numbers r_n,

$$1,\ -1,\ 7,\ 17,\ 5983,\ \ldots,$$

such that $$r_{n+1} = 2r_n^{\,2} - 3^{2^{n-1}},$$

and take the series of residues to the modulus B. The number B is prime if the first zero residue has a position comprised between $2q$ and $4q+1$; it is composite if no one of the $4q+1$ first residues is equal to zero; and, if $\alpha, =2q,$ is the position of the first zero residue, the divisors of B belong to the linear form $2^{\alpha}k \pm 1$, combined with those of the quadratic divisors of the form $2x^2 - y^2$.

Thus $r_8 = 5983 = 193 \times 31$; therefore $2^5 - 1$ is prime.

MATHEMATICAL NOTES.

A Problem in Partitions.

Take for instance 6 letters; a partition into 3's, such as *abc*, *def* contains the 6 duads *ab*, *ac*, *bc*, *de*, *df*, *ef*. A partition into 2's such as *ab.cd.ef* contains the 3 duads *ab*, *cd*, *ef*. Hence if there are α partitions into 3's, and β partitions into 2's, and these contain all the duads each once and only once, $6\alpha + 3\beta = 15$, or $2\alpha + \beta = 5$. The solutions of this last equation are ($\alpha = 0$, $\beta = 5$), ($\alpha = 1$, $\beta = 3$), ($\alpha = 2$, $\beta = 1$), and it is at once seen that the first two sets give solutions of the partition problem, but that the third set gives no solution; thus we have

$\alpha = 0,\ \beta = 5$	$\alpha = 1,\ \beta = 3$
ab . cd . ef	*abc . def*
ac . be . ef	*ad . be . cf*
ad . bf . ce	*ae . bf . cd*
ae . bd . cf	*af . bd . ce.*
af . bc . de	

Similarly for any other number of letters, for instance 15; if we have α partitions into 5's and β partitions into 3's, then if these contain all the duads $4\alpha + 2\beta = 14$, or what is the same $2\alpha + \beta = 7$; if $\alpha = 0$, $\beta = 7$, the partition problem can be solved (this is in fact the problem of the 15 school-girls), but can it be solved for any other values (and if so which values) of α, β? Or again for 30 letters; if we have α partitions into 5's, β partitions into 3's and γ partitions into 2's; then if these contain all the duads $4\alpha + 2\beta + \gamma = 29$; and the question is for what values of α, β, γ, does the partition-problem admit of solution.

The question is important from its connexion with the theory of groups, but it seems to be a very difficult one.

I take the opportunity of mentioning the following theorem: two non-commutative symbols α, β, which are such that $\beta\alpha = \alpha^2\beta^2$ cannot give rise to a group made up of symbols of the form $\alpha^p\beta^q$. In fact, the assumed relation gives $\beta\alpha^2 = \alpha^2\beta\alpha^2\beta^2$; and hence if $\beta\alpha^2$ be of the form in question, $= \alpha^x\beta^y$ suppose, we have $\alpha^x\beta^y = \alpha^2.\alpha^x\beta^y.\beta^2, = \alpha^{x+2}\beta^{y+2}$; that is $1 = \alpha^2\beta^2$, and thence $\beta\alpha = 1$, that is $\beta = \alpha^{-1}$, viz. the symbols are commutative, and the only group is that made up of the powers of α.

A. Cayley.

New Demonstration of the Fundamental Property of Linear Differential Equations.

The following demonstration is shorter than that which I have given in vol. IV., pp. 177, 178 (April 1875). Let

$$D^n y + A_1 D^{n-1} y + A_2 D^{n-2} y \ldots + A_{n-1} D y + A_n y = 0$$

be a linear differential equation, the coefficient being any functions whatever of x, and let z be a solution of this equation. Put $y = z\int t\,dx$ and the transformed equation in t will be, as is known from a remark due to Alembert, a linear equation of the order $n-1$ in t, viz.

$$D^{n-1} t + B_1 D^{n-2} t + B_2 D^{n-3} t \ldots + B_{n-2} D t + B_{n-1} t = 0.$$

Herein put $t = uz^{-1}$ or $u = zt$. The transformed equation in u will be of the $(n-1)^{\text{th}}$ order in u, viz.

$$D^{n-1} u + C_1 D^{n-2} u + C_2 D^{n-3} u \ldots + C_{n-1} D u + C_{n-1} u = 0.$$

Now, from the definition of t,

$$Dy = zt + Dz.\int t\,dx = u + \frac{Dz}{z} y,$$

whence
$$u = Dy - \frac{Dz}{z} y.$$

It follows that we have identically

$$D^n y + A_1 D^{n-1} y + A_2 D^{n-2} y \ldots + A_{n-1} D y + A_n y$$
$$= (D^{n-1} + C_1 D^{n-2} + C_2 D^{n-3} \ldots + C_{n-2} D + C_{n-1}) (Dy - ay) \ldots (1),$$

if
$$Dz - az = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Reciprocally, from (1) and (2) we can deduce that z is a solution of the given equation; in fact, since the latter may be written in the form

$$(D^{n-1}+C_1D^{n-2}\ldots+C_{n-1})\left(Dy-\frac{Dz}{z}y\right)=0,$$

if (1) and (2) hold good simultaneously, we see that it is satisfied by $y=z$.

This proposition, with that of which it is the reciprocal, constitute the fundamental property of linear equations.

PAUL MANSION.

Antwerp.

On Sylvester's Kinematic Paradox.

In his Royal Institution lecture,* Sylvester showed how by a linkage of 78† bars, the following problem termed by him the kinematic paradox might be solved: "Required to construct a link work fixed or centered at two of its points, such that (when the machine is set in motion) some other point or points therein shall be compelled to move in the line of centres."

Such a linkage may be very easily obtained by means of the relations connecting six points A, E, D, C, F, B, lying in order on a straight line, and such that

$$AB.AC=a^2,\ BC.BD=4a^2,\ EB.ED=a^2,\ FA.FE=2a^2.$$

For, let $AB=a\dfrac{x+a}{x-a}$, then $AC=a\dfrac{x-a}{x+a}$, $BC=\dfrac{4a^2x}{x^2-a^2}$,

$$BD=\frac{x^2-a^2}{x}=x-\frac{a^2}{x},\ EB=x,$$

$$AE=a\frac{x+a}{x-a}-x=\frac{2a^2}{x-a}-(x-a),$$

whence $FE=x-a$, and therefore $FB=a$, and is therefore a constant quantity.

If, therefore, A, B, C be connected by a reciprocator whose modulus is a^2; B, C, D by a reciprocator of modulus $4a^2$; E, B, D by a reciprocator of modulus a^2, and F, A, E by a reciprocator of modulus $2a^2$, then in the free motion of

* January 23, 1874, published in the Proceedings of the Royal Institution.

† It should be noticed, that by the use of four-bar reciprocators instead of Peaucellier's cells, the number 78 can be reduced to 52.

this linkage, consisting of only 16 bars, F and B, although not rigidly connected, will always remain at the same distance a apart.

This result I stated at the meeting of the London Mathematical Society on May 10, 1877 (see p. 55 of the present volume).

HARRY HART.

R. M. Academy, Woolwich.

A Theorem in Kinematics.

The following remarkable theorem may readily be deduced from that of Mr. Leudesdorf, *ante* p. 125, on which I have already at p. 165 made some brief observations.

"If one plane sliding upon another start from any position, move in any manner, making any number of rotations, and return to its initial position, then a circle can be found on the moving plane, every point on which describes on the fixed plane a curve of area 0; and if any other circle concentric with this zero-circle be taken on the moving plane, the areas described by all the points on this circle are the same, and are proportional to the area enclosed between that circle and the zero-circle.

"If the moving plane returns to its initial position without having made a complete rotation, the system of concentric circles is replaced by a system of parallel straight lines, the area described by a point on any line being proportional to the distance of that line from the zero line."

The existence of so singular a point as the centre of the concentric circles in the first part of this theorem is noteworthy.

A. B. KEMPE.

Western Circuit,
March 30th, 1878.

I. *Arithmetical Note.*

Write down a 5, divide it by 2 giving 2 with 1 over, divide 12 by 2 giving 6, divide 6 by 2 giving 3, divide 3 by 2 giving 1 with 1 over, divide 11 by 2 giving 5 with 1 over, divide 15 by 2 giving 7 with 1 over, and so on till the figures repeat. We thus obtain the figures 526315789473684 21, and these with a cipher prefixed are the period of $\frac{1}{19}$, viz.

$$\tfrac{1}{19} = \cdot\dot{0}5263157894736842\dot{1}.$$

If we start with 50 and halve in the same manner, prefixing two ciphers, we obtain the period of $\frac{1}{199}$, viz.

$$\tfrac{1}{199} = \cdot\dot{0}050251256281407035175879396984924623115577889 4$$

$$4723618090452261306532663316582914572864321608040\dot{2}01 \text{ (dot on final 1)}$$

Similarly, If we start with 500 and halve as before, we obtain, after prefixing three ciphers,

$$\tfrac{1}{1999} = \cdot\dot{0}005002501250625312656328164082041020510\ldots,$$

and, generally, the process gives the reciprocal of 1 followed by any numbers of 9.

If we start with 20, 200, 2000, &c., and divide continually by 5 instead of by 2, prefixing one, two, three, &c., ciphers, we obtain the periods of 49, 499, 4999, ... For example,

$$\tfrac{1}{49} = \cdot\dot{0}20408163265306122448979591836734693877555\dot{1}$$

$$\tfrac{1}{499} = \cdot\dot{0}02004008016032064128256513026052104208 4\ldots.$$

The process is very expeditious, the figures of the periods being obtained as fast as the hand can write them.

II. *Euler's Formula in Trigonometry.*

Take a chord AB (fig. 18), in a circle subtending an angle 2θ at the centre O, and therefore an angle θ at the circumference, so that $\angle ACB = \theta$. Bisect the arc AB in M, then $\angle ABM = \angle ACM = \frac{1}{2}\theta$, and the chords AM, MB are together equal to $AB \sec\frac{1}{2}\theta$. Similarly, if AM, MB be bisected in N_1, N_2, the four equal chords AN_1, N_1M, MN_2, N_2B are together equal to $AB \sec\frac{1}{2}\theta \sec\frac{1}{4}\theta$. Proceeding in this way, we see that if A, B be joined by 2^n equal chords inscribed in the arc AB, then the perimeter of these chords

$$= AB \sec\tfrac{1}{2}\theta \sec\tfrac{1}{4}\theta \ldots \sec\frac{1}{2^n}\theta.$$

Thus the ratio that the chord AB bears to the arc AB

$$= \cos\tfrac{1}{2}\theta \cdot \cos\tfrac{1}{4}\theta \cos\tfrac{1}{8}\theta \ldots,$$

where 2θ is the angle subtended by AB at the centre, and thus

$$\frac{\sin\theta}{\theta} = \cos\tfrac{1}{2}\theta \cos\tfrac{1}{4}\theta \cos\tfrac{1}{8}\theta \ldots.$$

It is, I think, interesting to notice how directly by the above process, the expression $a \sec\frac{1}{2}\theta \sec\frac{1}{4}\theta \sec\frac{1}{8}\theta \ldots$ admits of geometrical interpretation; but the process is, of course, not new; in fact, Vieta, Van Ceulen, and others who used the

method of polygons were in effect acquainted with Euler's formula, though it did not then admit of expression in its general form, as the cosine had not at that time been introduced.

III. *Numerical value of a Series.*

The value of the series

$$\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{11^2}+\frac{1}{13^2}+\text{\&c. } ad\ inf.$$

(the denominators being the squares of the prime numbers), is, to 24 places,

$$0{\cdot}4522\,4742\,0041\,0654\,9850\,6546,$$

the last figure being uncertain.

Euler, in the *Introductio in Analysin Infinitorum*, t. I. §282, gives the value to 15 places as

$$0{\cdot}4522\,4742\,0041\,222,$$

so that his last three figures are erroneous. As far as I know, Euler's calculation has not hitherto been verified or extended.

J. W. L. GLAISHER.

TRANSACTIONS OF SOCIETIES.

London Mathematical Society.

Thursday, February, 14th, 1878.—Lord Rayleigh, F.R.S., *President*, and subsequently Mr. C. W. Merrifield, F.R.S., *Vice-President*, in the chair. The following communications were made to the Society: "On a General Method of Solving Partial Differential Equations," Prof. H. W. Lloyd Tanner; "On the Conditions for Steady Motion of a Fluid," Prof. H. Lamb, Adelaide; "On a Property of the Four-piece Linkage, and on a curious Locus in Linkages," Mr. A. B. Kempe, B.A.; "On Robert Flower's Method of computing Logarithms," Mr. S. M. Drach; "On the Pluckerian characteristics of the Modular Equations," Prof. H. J. S. Smith, F.R.S. Mr. Drach also exhibited drawings of Tricircloids made several years ago for Mr. Perigal.

Thursday, March 14th, 1878.—Lord Rayleigh, F.R.S., *President*, in the chair. Mr. Artemas Martin, M.A., Erie, Pa, was proposed for election. The Hon. Sec. read portions of a paper by Prof. J. Clerk Maxwell, F.R.S., "On the Electrical capacity of a long narrow cylinder and of a disc of sensible thickness." Prof. Cayley, Mr. J. W. L. Glaisher, Mr. S. Roberts, and the President, made short communications to the Society.

R. TUCKER, M.A., *Hon. Sec.*

THE END OF VOL. VII.

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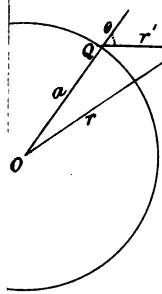
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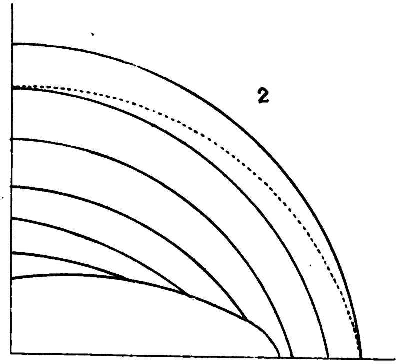

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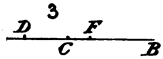
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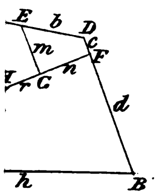
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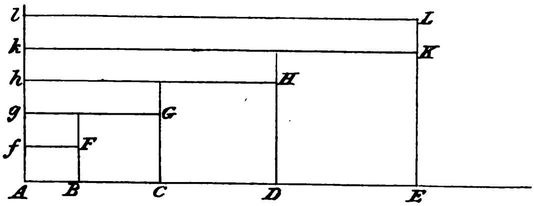
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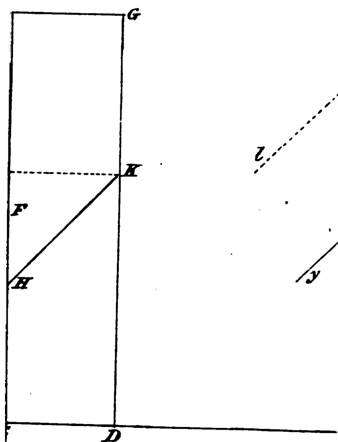
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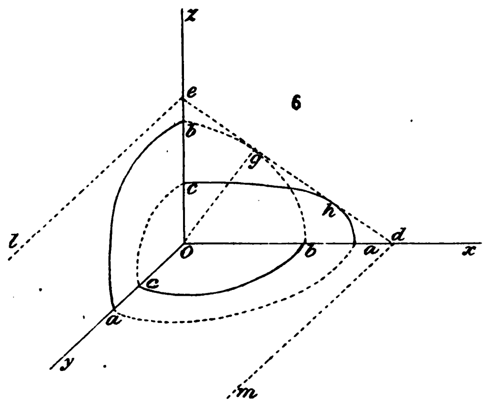
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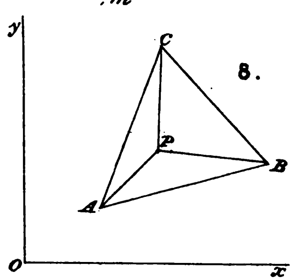
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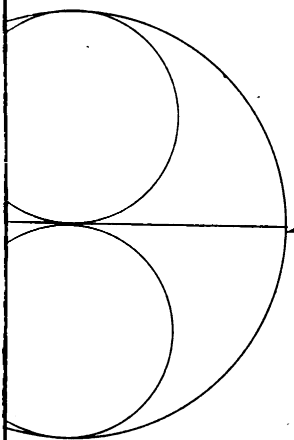

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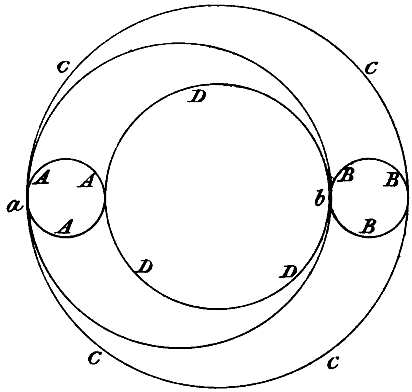

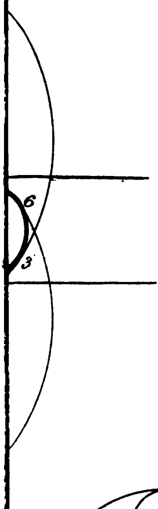

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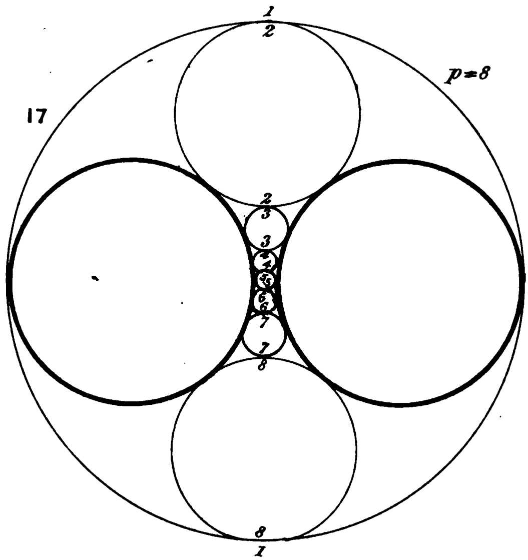

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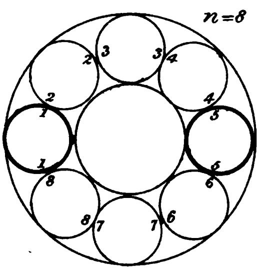

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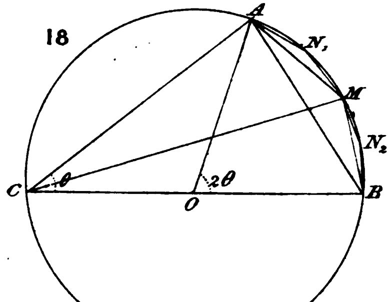

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